COUPLED CELL NETWORKS - INTERPLAY BETWEEN ARCHITECTURE & DYNAMICS

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Doctor of Philosophy

By

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COUPLED CELL NETWORKS - INTERPLAY BETWEEN ARCHITECTURE & DYNAMICS

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Abstract

We are interested in studying the dynamics on coupled cell networks using the network topology or architecture. The original motivation of this work comes from the work by Aguiar *et al.* [4] on dynamics of coupled cell networks and Dias and Stewart [18] on equivalence of coupled cell networks with linear phase space. In this thesis, we give a necessary and sufficient condition for the dynamical equivalence of two coupled cell networks. The results are applicable to both continuous and discrete dynamical systems and are framed in terms of what we call input and output equivalence. We also give an algorithm that allows explicit construction of the cells in a system with a given network architecture in terms of the cells from an equivalent system with different network architecture. We provide a number of examples to illustrate the results we obtain.

The dynamics of large coupled dynamical systems can be extremely difficult to analyze. One way of approaching the study of dynamics of networks is to start with a simple, well-understood but interesting small network and investigate how the simple network can be naturally embedded in a larger network in such a way that the dynamics of the small network appears in the dynamics of the larger network. This process is termed *inflation*, and was introduced by Aguiar *et al.* [4]. We give a necessary and sufficient condition for the existence of a strongly connected inflation. We also provide a simple algorithm to construct the inflation as a sequence of simple inflations.

Finally, we give an example of a 3-cell coupled cell network having non-trivial synchrony subspaces that supports heteroclinic cycles with switching between them.

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Chapter 1

Introduction

Networks are used as models in a wide range of applications in biology, physics, chemistry, engineering, and the social sciences (for many characteristic examples, we refer to the survey by Newman [40]). Of particular interest, especially in biology and engineering, are networks of interacting dynamical systems. In the last two decades, enormous work has been done on networks – analyzing the behavior of the network by changing the coupling strength or, the connection structures, analyzing the synchrony patterns, etc [23, 31, 10, 11, 46, 17, 20, 25, 6, 7, 26, 24, 8, 39, 45]. The network structures are quite different in various fields. When we say network structure, we mean the number or type of nodes, the types of edges or connections – static or dynamic, presence of noise or delay in the network, etc. Following the work of Kuramoto on networks of coupled phase oscillators with all-to-all coupling [38], there has been an intensive effort to understand the dynamical behavior of networks in terms of invariants of the network and to find conditions that imply the emergence of synchronization in complex networks. Typically the methods used for large complex networks are statistical and allow for the interaction of structurally identical units which have parameters (for example, coupling strength) and network connections distributed according to a statistical law. For a characteristic illustration of this approach, we refer to the article by Restrepo *et al.* [31] where conditions on the adjacency matrix of a network are shown to lead to synchronization of a network. The eigenvalues of the network adjacency matrix play an important role in understanding the network dynamics such as synchronization, linear stability of the equillibria or synchronized states. Restrepo *et al.* [32] develop approximations to the largest eigenvalue which plays a key role in the dynamics. The adjacency matrix is an essential invariant of the network.

In a rather different direction, methods from symmetric dynamics have been used to understand symmetrically coupled networks of *identical* oscillators. One of the early works in this area is due to Ashwin & Swift [12] who study the dynamics of weakly coupled identical oscillators. More recently, Stewart, Golubitsky, and coworkers [23, 24, 26] have formulated a general theory for networks of interacting (typically, identical) dynamical systems (for a overview, see [25]). Typically no symmetry is assumed for the network architecture though local symmetries may be present and these are described using groupoid formalism. While this type of model is unlikely to apply exactly to large biological networks, such as neuronal networks, it is possible that the assumption of identical dynamical system may be applicable to an 'averaged' network – that is, after the addition of noise and in the regime where there is synchronization. In any case, small (asymmetric) networks of identical dynamical systems typically display interesting and often quite non-generic dynamics [4, 25] and there is the significant question concerning the extent to which the large networks that occur in say biology or engineering applications can be modelled in a hierarchical way as a network of small networks or *motifs* [9]. Natasa [42] developed tools to decode the large network data sets to understand the biological processes. They observe that local node similarity corresponds to similarity in biological function and involvement in diseases. Network theory applies to social networks as well. Social networks are often transient, and evolve over time. The relationship between friends may change with time, so the edges between various nodes in the social networks are dynamic and change with time. Peter Grindrod *et al.* [27] consider networks with memory dependent edges, and analyze the long term asymptotic behavior as a function of parameters controlling the memory dependence of the edges. They show that either the networks continue to evolve forever or become static, which may contain dead or extinct edges.

In this work one of the questions we focus on is when two networks with apparently quite different topologies or architectures are dynamically equivalent (we give the precise definition in Chapters 3, 4, 5 – we do not mean topological equivalence in the sense of conjugacy). We also consider and solve the problem of the explicit realization of equivalence for continuous dynamics and give a partial solution for discrete dynamics. These results are probably of greatest interest for relatively small networks. Indeed, although the invariants we describe are quite simple, they depend on the ordering of the cells and so checking of the equivalence of two networks potentially requires consideration of many different orderings. We emphasize that the methods we use involve precise descriptions of dynamics and are not statistical. The approach to networks we use in this work is synthetic and combinatorial in character. In particular, we adopt a simple and transparent 'flow-chart' formalism similar to that used in electrical and computer engineering. Indeed, ideas from analog computation motivate parts of our approach to networks (for more background, we refer to [20, 4]).

We view a network as a collection of interacting dynamical systems or 'cells'. The dynamics of a cell will be deterministic (specified by a vector field – continuous dynamics; or map – discrete dynamics). A *coupled cell system* will then be a specific set of individual but interacting cells. Each cell will have an *output* and a number of *inputs* coming from other cells in the system. An output might be the *state* of the cell (that is, the point in the phase space of the dynamical system which determines the evolution of the cell) or it could be a scalar or vector valued observable (for example, temperature and pressure or a membrane potential). Our setup is robust enough to handle both situations. Typically, if the network is small and cell dynamics are low dimensional, we assume the output is the state of the cell. For large networks or high dimensional cell dynamics, a vector valued observable is likely to be more appropriate. We emphasize that each cell has only one (type of) output. However, the output may be connected to many different inputs; these inputs do not have to be of the same type. (An analogy is having several different appliances powered by the same power socket.) Given a collection of cells there may be many different ways of connecting them into a network and the combinatorics of this process – viewed in a general way that allows groups of appropriately connected cells to define new types of cell – is one of the issues that is addressed here.

A coupled cell system has a *network architecture* or *network structure* that can be represented by a directed graph with vertices corresponding to cells and each directed edge corresponding to a specific output–input connection. Different input types will correspond to different edge types in the graph. We use the term *coupled cell network* to refer to a network of coupled cells with a specified network architecture. Thus vertices correspond to cells, edges to connections. If we want to emphasize a specific coupled cell network, we often use the term *coupled cell system*.

Suppose that $\mathcal{M}$ and $\mathcal{N}$ are coupled cell networks which both have n cells. Assume that cells are modelled by vector fields (that is, continuous dynamics determined by local solution of ordinary differential equations) and that the phase space of each cell is a smooth manifold. We write $\mathcal{M} \prec \mathcal{N}$ ($\mathcal{M}$ is *dominated* by $\mathcal{N}$) if for every coupled cell system $\mathcal{F}$ with architecture $\mathcal{M}$, there exists a coupled cell system $\mathcal{F}^\star$ with architecture $\mathcal{N}$ such that $\mathcal{F}$ and $\mathcal{F}^\star$ have identical dynamics. This definition simply means that the dynamics of any system $\mathcal{F}$ with network architecture $\mathcal{M}$ can be realized by a system $\mathcal{F}^\star$ with network architecture $\mathcal{N}$. Implicit in the definition is the requirement that there is a correspondence between the cells of $\mathcal{F}$ and $\mathcal{F}^\star$ so that corresponding cells have identical phase space. We regard $\mathcal{M}$ and $\mathcal{N}$ as *equivalent* networks if $\mathcal{M} \prec \mathcal{N}$ and $\mathcal{N} \prec \mathcal{M}$. We may similarly define equivalence for networks modelled by discrete dynamics. Our main result (Theorem 4.2.6) gives a simple necessary and sufficient condition for the equivalence of two networks in terms of an invariant that depends only on the network architecture (specifically, only on the adjacency matrices of the network). This result generalizes the linear equivalence results of Dias and Stewart [18] but the conceptual approach and methods are quite different. We emphasize that the algebraic condition, formulated in terms of adjacency matrices, is simple and easy to check –

at least if we are given an ordering of the cells. However, realizing the equivalence using an output or input equivalence is by no means straightforward, especially when there are symmetric inputs (see Chapter 5). The second question we address concerns the relation $\mathcal{M} \prec \mathcal{N}$. If the coupled cell system $\mathcal{F}$ has architecture $\mathcal{M}$, we present an explicit algorithm that allows us to construct a coupled cell system $\mathcal{F}^\star$ with architecture $\mathcal{N}$ such that $\mathcal{F}^\star$ has identical dynamics to $\mathcal{F}$. *The cells in the coupled cell system $\mathcal{F}^\star$ are constructed using the cells of $\mathcal{F}$ together with some passive cells that either scale or add and subtract outputs or inputs.* We also give a number of equivalence results that hold for discrete dynamics and for various classes of phase space (such as connected Abelian groups). In order to carry out this program we introduce the ideas of *input* and *output* equivalence. The idea of input equivalence is motivated by linear systems theory and involves taking linear combinations of inputs (necessarily, cell outputs must be either vector valued observables or state spaces must be linear). Output equivalence is formulated in terms of linear combinations of outputs or linear combinations of vector fields. Output equivalence may or may not apply to discrete systems defined on nonlinear spaces but it does apply to discretizations of ordinary differential equations.

We now briefly describe the relation of our work to earlier results in this area. As we indicated above, a general theory of networks of coupled cells has been developed by Stewart, Golubitsky and coworkers. Their approach is relatively algebraic in character and strongly depends on groupoid formalism, graphs, and the idea of a quotient network. Dias and Stewart [18] define equivalence in this setting and prove results on equivalence of networks when the phase space is a vector space. We provisionally refer to the definition of Dias and Stewart as 'functional equivalence'. It is easy to see that functional equivalence implies dynamical equivalence. The converse is also true since dynamical equivalence implies dynamical equivalence with phase space $\mathbb{R}$. Using the results of Dias and Stewart [18], dynamical equivalence with phase space $\mathbb{R}$ implies linear equivalence which implies functional equivalence. The methods of Dias and Stewart do not apply to systems for which the phase space is a general manifold nor do their methods give algorithms for realizing the equivalence. On account of their use of invariant theory and Schwarz' theorem on smooth invariants, they assume the phase space is linear and maps are smooth (that is C^∞).

Our work makes the following contribution to the problem of classification of networks based on their dynamics. It introduces methods that extend the notion of linear equivalence described in [18]. While our condition is similar to their condition on adjacency matrices, it is more general in the sense that it allows the phase space to be a general manifold. Moreover, our results give an algorithmic construction for obtaining equivalences using input or output equivalence. The methods we present make no use of smooth invariant theory and indeed give a relatively elementary proof of the main results in [18].

1.1 Overview of thesis

In Chapter 2, we introduce the basic definitions and notational conventions needed for the work. Our aim is to get the language and results as transparent as we can and, as far as possible, hide the (notational) complexities in the proofs of the results.

There are two aspects of networks discussed in the thesis – one relates to the study of dynamics on the network using the architecture or the topology of the network, the other is the network architecture itself, ignoring the dynamical structure on it. We find conditions on the adjacency matrices to study the dynamics on the network. Also, we find conditions on the adjacency matrices to construct large networks that contains the dynamics of a small network on an invariant subspace.

In Chapter 3, we introduce the concept of dynamical equivalence of coupled cell networks. In Chapter 4, we define the notion of input and output equivalence for networks with asymmetric inputs which captures the dynamical behavior of the network using the essential invariant of the network – the *adjacency matrix*. We then prove the main theorems on dynamical equivalence. Although equivalence always implies output equivalence for continuous systems, this is *not* always the case for input equivalence. In Chapter 5, we extend the notion of input and output equivalence for general networks and give necessary and sufficient conditions for equivalence to imply input equivalence. In Chapters 4, 5, we also present examples that illustrate some outstanding issues for discrete networks.

In Chapter 6, we give the necessary and sufficient condition for the existence of a strongly connected inflation of a network. We first establish the basic properties of inflation and simple inflation for strongly connected networks as well as giving some simple examples. We then state and prove our main result for the case of networks with one edge type. Then we provide the straightforward extension to general networks and multiple input types. As a consequence of our constructions, we obtain an algorithm for the construction of a strongly connected inflation. As the main result in this chapter is graph theoretic in character, we will not need to discuss the dynamical structure on cells. Thus our focus in this chapter will be on the network architecture and associated graph of the coupled cell network.

The results in Chapter 4 and 5 are joint work with my advisor, Mike Field and appears in [2, 3], Chapter 6 is based on [1]. The parts of this thesis which do not appear in [2, 3, 1], are also done under the constant guidance of my advisor.

Chapter 2

Background and Preliminaries

2.1 Generalities on coupled cell networks

We distinguish between a *coupled cell network*, an abstract arrangement of cells and connections, and a *coupled cell system* which is a particular realization of a coupled cell network as a system of coupled dynamical equations. If we wish to emphasize the network graph rather than the dynamic structure, we refer to the *network architecture*. We shall be particularly interested in networks with specific network architecture which satisfy additional constraints. Typically these constraints will relate to either the phase space or the type of input or output or the type of dynamics (for example, continuous or discrete).

2.1.1 Structure of coupled cell networks: cells and connections

For the moment we regard a cell as a ‘black box’ that admits various types of input (from other cells) and which has an output which is uniquely determined by the inputs and the initial state of the cell. The output may vary in discrete or continuous time. Two cells are regarded as being of the same *class* or *identical* if the same inputs and initial state always result in the same output. We will largely restrict to networks of *identical* cells and leave the simple and straightforward extensions to more general networks containing different types of cell to the remarks (see also [20, 4]). For clarity, we always use the word *class* in the sense used above: ‘two cells are of the same (or different) class’. We restrict the use of the word *type* to distinguish inputs of a cell.

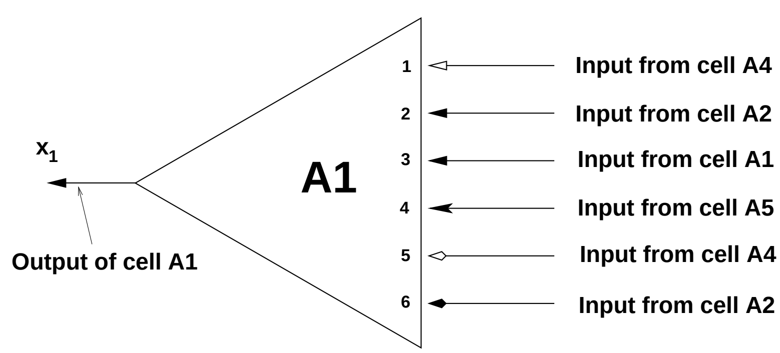


Figure 2.1: A cell with six inputs and one output. Inputs of the same type (for example the second and third input) can be permuted without affecting the output.

Figure 2.1 shows a cell, labelled **A1**, which accepts six inputs from cells **A1**,

A2, **A4**, and **A5**. We assume the cells **A2**, **A4**, and **A5** are of the same class as **A1**. We denote the output of **A1** by x_1 and regard x_1 as specifying the *state* of the cell **A1** (later we shall vary this definition of output). Generally we do not regard inputs as interchangeable and we may distinguish different types of input by, for example, using different arrow heads. Referring to the figure, inputs 2 and 3 are of the *same type* whereas inputs $1, 4, 5$, and 6 are of different types and of different type from inputs 2 and 3: the cell has five distinct input types. We can interchange inputs 2 and 3 without changing the behaviour of the cell. We refer to these inputs as *symmetric*. If there are no pairs of symmetric inputs, we say that the cell has *asymmetric inputs*.

We may think of cells as being coupled together using 'patchcords'. Each patchcord goes from the output of a cell to the input of the same or another cell. We show two simple examples using identical cells in Figure 2.2.

When all the inputs of all the cells are filled, as they are in Figure 2.2(ii), we refer to the set of cells and connections as a *network of coupled cells*.

We now give a formal definition of a coupled cell network based on the approach in Aguiar *et al.* [4].

Definition 2.1.1. A *coupled (identical) cell network* $\mathcal{N}$ consists of a finite number of identical cells such that

(a) The cells are patched together according to the input-output rules described above.

(b) There are no unfilled inputs.

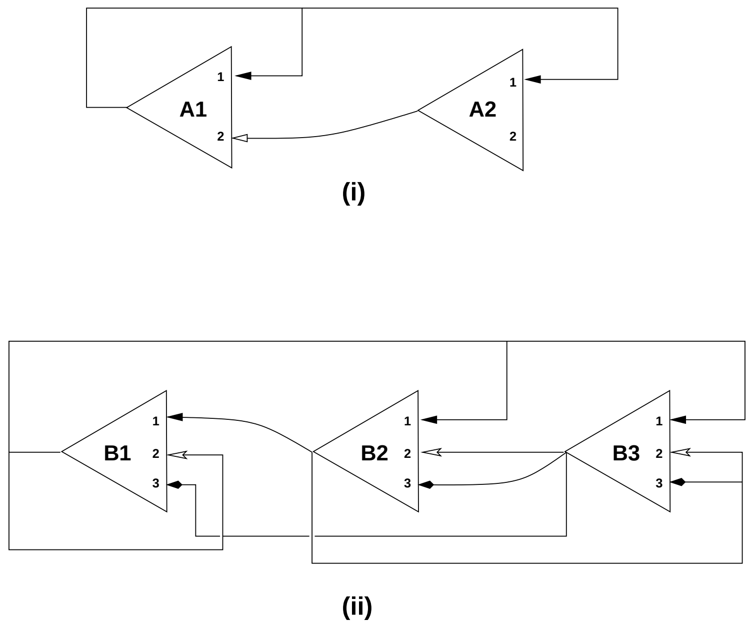


Figure 2.2: Examples of coupled cells: (i) shows an incomplete network with an unfilled input cell **A2**, (ii) shows a three cell asymmetric input network; all inputs are filled.

Remarks 2.1.2. (1) We always assume cells have at least one input.

(2) There are no restrictions on the number of outputs from a cell.

(3) If a cell has multiple inputs of the *same* type (that is, symmetric inputs), it is immaterial which input of the symmetric set the patchcord is plugged into. More precisely, if a cell **A** in the network has $k > 1$ inputs of the same type, then permutation of the k connections to these inputs is allowed and will not change the network structure. When it comes to graphical representation of networks, we always represent input types to cells in the same order. If there are symmetric inputs, these are always grouped together (as in Figure 2.1).

(4) A coupled cell network determines an associated directed graph (the *network*

architecture) where the vertices of the graph are the cells and there is a directed edge from cell $\mathbf{X}$ to cell $\mathbf{Y}$ if and only if cell $\mathbf{Y}$ receives an input from cell $\mathbf{X}$. Different input types will correspond to different edge types in the graph. If there are p different input types, then there will be p different edge types in the associated graph.

We let $\mathbb{Z}$ denote the integers, $\mathbb{Z}^+$ denote the non-negative integers, $\mathbb{N}$ denote the strictly positive integers, and $\mathbb{Q}$ the rational numbers. If $k \in \mathbb{N}$, we use the abbreviated notation $\mathbf{k} = \{1, \cdots, k\}$, $\overline{\mathbf{k}} = \{0, 1, \cdots, k\}$. If $p \in \mathbb{N}$, then $\mathbf{k}^p$ denotes the set of all p-tuples $(k_1, \cdots, k_p)$, where $k_j \in \mathbf{k}$, for all $j \in \mathbf{p}$.

Adjacency matrices of a network

As we shall see in the following chapters, the key invariant of a coupled cell network is defined using the set of *adjacency* matrices. We recall the definition[1] appropriate to our context. Let $\mathbf{A}$ be a cell class. We suppose that $\mathbf{A}$ has r inputs and p input types. Let $\mathbf{A}$ have r_ℓ inputs of type ℓ, for $\ell \in \mathbf{p}$. Necessarily $r_1 + \cdots + r_p = r$. Of course, we assume $r_\ell \geq 1$. The cell $\mathbf{A}$ has asymmetric inputs iff $p = r$ and then $r_\ell = 1$, $\ell \in \mathbf{p}$. Suppose that $\mathcal{N}$ is a coupled cell network consisting of n cells $C_1, \cdots, C_n$ each of class $\mathbf{A}$. We define $n \times n$ matrices $N_0, \cdots, N_p$. We take N_0 to be the identity matrix[2]. For $\ell \in \mathbf{p}$, we let $N_\ell = [n_{ij}^\ell]$ be the matrix defined by $n_{ij}^\ell = k$ if there are exactly k inputs of type ℓ to C_j from the cell C_i. If there are no inputs of type ℓ from C_i, then $k = 0$. We refer to N_ℓ as the *adjacency matrix of type* ℓ for

[1] Conventions vary. We choose the definition commonly used in graph theory; others take the transpose of the matrices we define.

[2] Strictly speaking, we only include N_0 if we allow internal variables – that is, the evolution of the state of the cell depends on its state, not just its initial state.

$\mathcal{N}$. Observe that the jth *column* of N_ℓ identifies the source cells for all the inputs of type ℓ to the cell C_j. If $\mathbf{A}$ has asymmetric inputs, then there are $r+1$ adjacency matrices, $N_0, \cdots, N_r$ and each adjacency matrix will be a $0-1$ matrix with column sum equal to 1. If there are symmetric inputs, then there will be $p+1 < r+1$ adjacency matrices. The column sum of N_ℓ gives the number of inputs of type ℓ to $\mathbf{A}$ and we refer to the column sum as the *valency* of N_ℓ. We denote the valency of N_ℓ by $\nu(\ell)$ and remark that $\nu(\ell) = \sum_{i=1}^{n} n_{ij}^{\ell} = r_\ell$ (independent of $j \in \mathbf{n}$). Let $\mathbb{A}(\mathcal{N})$ denote the (ordered) set $\{N_0, \cdots, N_p\}$ of adjacency matrices of $\mathcal{N}$.

Remarks 2.1.3. (1) If we allow multiple cell classes, then we define an adjacency matrix for each input type of every cell class.
(2) If we deny self-loops in the network structure, then the diagonal entries of the adjacency matrices $N_1, \cdots, N_p$ will all be zero.

The adjacency matrices N_1, N_2, N_3 for the network of Figure 2.2(ii) are shown below.

$$N_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ N_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ N_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Connection Matrix

Networks with asymmetric inputs

Assume for the moment that $\mathcal{M}$ is a coupled cell network consisting of n identical cells with r *asymmetric inputs* (the number r of inputs equals the number of input types p). Label the cells as $\mathbf{C}_1, \cdots, \mathbf{C}_n$. For each cell $\mathbf{C}_j$, we let $\mathfrak{m}^j = (\mathfrak{m}_1^j, \cdots, \mathfrak{m}_r^j) \in \mathbf{n}^p$ denote the r-tuple defined by requiring that there is an output from $\mathbf{C}_{\mathfrak{m}_i^j}$ to input i of $\mathbf{C}_j$. That is, $\mathfrak{m}_i^j$ identifies the source cell for the input of type i to C_j. If we denote the adjacency matrices of $\mathcal{M}$ by $M_0, \cdots, M_r$, then $\mathfrak{m}_i^j$ is the row index of the unique non-zero entry in column j of M_i. The $r \times n$ matrix $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ specifies the complete set of all connections to the cells in $\mathcal{M}$. We refer to $\mathfrak{m}$ as the *connection matrix* of the network $\mathcal{M}$. We emphasize that in order to define uniquely the connection matrix (and adjacency matrices) of a network, we need to order the cells and the input types. For future reference, note that if V is any vector space, then

$$\sum_{i=1}^{n} m_{ij}^{0} x_i = x_j, \qquad \sum_{i=1}^{n} m_{ij}^{\ell} x_i = x_{\mathfrak{m}_\ell^j},\ \ell \in \mathbf{r},\ j \in \mathbf{n}, \tag{2.1.1}$$

where $x_1, \cdots, x_n \in V$ and $M_\ell = [m_{ij}^\ell]$, $\ell \in \{0, \cdots, r\}$, are the adjacency matrices. The result is obvious if $\ell = 0$, so suppose $\ell \in \mathbf{r}$. Then $m_{\alpha j}^\ell \neq 0$ iff C_j has an input of type ℓ from C_α. If this is so, then $m_{\alpha j}^\ell = 1$ and $m_{ij}^\ell = 0$, $i \neq \alpha$ (since we assume asymmetric inputs). Hence $\sum_{i \in \mathbf{n}} m_{ij}^\ell x_i = m_{\alpha j}^\ell x_\alpha = x_\alpha = x_{\mathfrak{m}_\ell^j}$, by definition of $\mathfrak{m}_\ell^j$.

General networks

We may also define a connection matrix for networks with symmetric inputs. Let $\mathcal{M}$ be a coupled cell network with n identical cells, r inputs and p input types. Suppose that there are r_i-inputs of type i, $i \in \mathbf{p}$ (if inputs are asymmetric then $r_i = 1$ for all i and $r = \sum_{i=1}^p r_i = p$). For $j \in \mathbf{n}$, we define $\mathfrak{m}_i^j \in \mathbf{n}^{r_i}$ by requiring that $\mathfrak{m}_i^j = (a_1, \cdots, a_{r_i})$, where $a_1, \cdots, a_{r_i} \in \mathbf{n}$ and a_s identifies the source cell for the sth input of type i to $\mathbf{C}_j$. The vector $\mathfrak{m}^j = (\mathfrak{m}_1^j, \cdots, \mathfrak{m}_p^j) \in \prod_{i=1}^p \mathbf{n}^{r_i} \cong \mathbf{n}^r$ specifies all of the inputs to $\mathbf{C}_j$. We say that the $r \times n$ matrix $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ is a connection matrix for the network. If, in addition, we require that $1 \le a_1 \le \cdots \le a_{r_i} \le n$, then $\mathfrak{m}_i^j$ is uniquely determined and we refer to the associated connection matrix as the *default* connection matrix of the network (or just *the* connection matrix of the network).

Example 2.1.4. 1. The default connection matrix for the network of Figure 2.2(ii) is $\mathfrak{m} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 3 & 2 \end{bmatrix}$.

2. The default connection matrix for a network with two types of inputs and adjacency matrices $N_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, $N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$ is $\mathfrak{m} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \\ 3 & 3 & 2 \end{bmatrix}$.

2.1.2 Discrete and continuous coupled cell systems

We now define two basic classes of coupled cell networks with specified network architecture. First some notational conventions. If $\mathcal{N}$ denotes a network architecture, then by $\mathcal{F} \in \mathcal{N}$ we mean that $\mathcal{F}$ is a coupled cell system with connection and input type structure given by $\mathcal{N}$. If $\mathcal{N}$ is a coupled cell network (viewed as the collection of all coupled cell systems with network architecture $\mathcal{N}$), then the number of cells $n = n(\mathcal{N})$, the number of input types $p = p(\mathcal{N})$, and the total number of inputs $r = r(\mathcal{N})$, are the same for all systems $\mathcal{F} \in \mathcal{N}$.

Continuous dynamics modelled by ordinary differential equations

For continuous dynamics, we assume that cell outputs (and therefore inputs) depend continuously on time. The standard model for this situation is where each cell is modelled by an (autonomous) ordinary differential equation. In this work, we always assume that the evolution in time of cells depends on their internal state (not just their initial state). The underlying phase space for a cell will be a smooth manifold M, often $\mathbb{R}^N$, $N \geq 1$, or $\mathbb{T}$ (the unit circle). We assume the associated vector field has sufficient regularity (say C^1) to guarantee unique solutions.

If the phase space for a cell $\mathbf{C}$ is M, then the output $x(t)$ of $\mathbf{C}$ at time t defines a smooth curve in M and $x(0)$ will be the initial state of the cell (at time $t = 0$). We identify $x(t)$ with the internal state of the cell. If the cell has no inputs, then the ordinary differential equation model for the cell is $x' = f(x)$, where f is a vector field on M.

Suppose that we are given a coupled cell system $\mathcal{F} \in \mathcal{M}$ with identical cells $\{\mathbf{C}_1, \mathbf{C}_2, \cdots, \mathbf{C}_n\}$. Assume that cells are modelled by ordinary differential equations and that each cell has r (asymmetric) inputs. For $j \in \mathbf{n}$, the dynamics of $\mathbf{C}_j$ will be given by a differential equation

$$x_j' = f(x_j; x_{\mathfrak{m}_1^j}, x_{\mathfrak{m}_2^j}, \cdots, x_{\mathfrak{m}_r^j}),$$

where $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ is the connection matrix of the network (see the previous section). Observe that we always write the internal variable x_j as the first variable of the vector field f. Since cells are assumed identical, the vector field f is independent of $j \in \mathbf{n}$. We often say that the dynamics of the system $\mathcal{F}$ is *modelled by*

f and refer to f as the *model* for $\mathcal{F}$. If we need to emphasize the dependence of the model f on the system $\mathcal{F}$, we write $f_{\mathcal{F}}$ rather than f. All of this terminology applies equally well to discrete systems (see below).

Remarks 2.1.5. (1) If there is no dependence on the internal variable, we omit the initial x_j and write $x_j' = f(x_{\mathfrak{m}_1^j}, x_{\mathfrak{m}_2^j}, \cdots, x_{\mathfrak{m}_r^j})$.
(2) We do not require that $\mathfrak{m}_1^j, \cdots, \mathfrak{m}_r^j$ are distinct integers; indeed they may all be equal. If there are symmetric inputs we group these together and designate the group by an overline. For example, if the vector field f is symmetric in the first k-inputs and asymmetric in the remaining inputs we write

$$f(x_j; \overline{x_{\mathfrak{m}_1^j}, \cdots, x_{\mathfrak{m}_k^j}}, x_{\mathfrak{m}_{k+1}^j}, \cdots, x_{\mathfrak{m}_r^j}).$$

Example 2.1.6. A coupled cell system with the network architecture of Figure 2.2(ii) is realized by the differential equations

$$\begin{aligned} x_1' &= f(x_1; x_2, x_1, x_3), \\ x_2' &= f(x_2; x_1, x_3, x_3), \\ x_3' &= f(x_3; x_1, x_2, x_2). \end{aligned}$$

where $f : M \times M^3 \rightarrow TM$ is a (smooth) family of vector fields on M, depending on parameters in M^3. That is, for each $(x, (y, z, u)) \in M \times M^3$, $f(x; y, z, u) \in T_xM$.

Discrete dynamics

We continue with the notation of the previous section. We define a discrete time coupled cell system by considering a system of coupled maps updated at regular time intervals.

Exactly as in the continuous time case, we model a cell $\mathbf{C}_j$ at time N using a phase space variable $x_j(N)$ and then update all cells simultaneously by

$$x_j(N+1) = f(x_j(N); x_{\mathfrak{m}_1^j}(N), x_{\mathfrak{m}_2^j}(N), \cdots, x_{\mathfrak{m}_r^j}(N)),\ j \in \mathbf{n}, \tag{2.1.2}$$

where f is a continuous or smooth function depending on the internal state $x(N)$ together with the r inputs to the cell.

Remarks 2.1.7. (1) The assumption of regular updates implies the existence of a synchronizing mechanism such as a clock. It is also possible (and very useful) to define *asynchronous* systems.
(2) The diagrammatic conventions we follow are reminiscent of the transfer function diagrams used in linear systems theory. However, we are working with nonlinear systems and not taking Laplace transforms. For discrete dynamics, we start at $t = 0$ with each cell initialized and then we update according to (2.1.2) at specific time increments $\delta > 0$. The case of continuous dynamics can be viewed as the limiting case as $\delta \to 0$ – indeed, ordinary differential equations are typically solved on a digital computer by replacing the continuous model by a discrete model with an appropriately small value of $\delta > 0$.
(3) In many cases it can be useful to combine both discrete and continuous dynamics in a coupled cell system. This is particularly so for models of fast-slow systems

(prevalent for example in neural systems). We may also consider systems with thresholds controlling switching in cells where the dynamics are, for the most part, governed by (smooth) differential equations. We refer to this type of system as a *hybrid* coupled cell system. Such systems are of theoretical interest and important in many applications. Recently, they have attracted the attention of control theorists and engineers. We refer to Aguiar et al [4] for definitions and some simple examples.

2.1.3 Passive cells

As we shall see in Chapters 4 and 5, it is sometimes useful to incorporate 'passive' cells into a network with continuous or discrete dynamics. We give two examples that we shall use later.

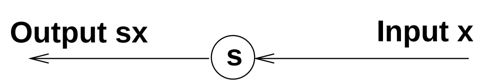


Figure 2.3: Scaling cell

Example 2.1.8 (Scale)**.** A *scaling* cell has one input. The input can either be a vector field or vector in $\mathbb{R}^N$. If the input is x then the output is sx, where $s \in \mathbb{R}$ is fixed. We use the notation shown in Figure 2.3 for a scaling cell.

Example 2.1.9 (Addition/Subtraction)**.** An *Add* cell has at least two inputs which must be either vector fields, points in $\mathbb{R}^N$ or, more generally, points in an Abelian Lie group. In Figure 2.4(a) we show an Add cell with four inputs. If the inputs are $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ then the output is $\mathbf{e}+\mathbf{f}+\mathbf{g}+\mathbf{h}$. We remark that an Add

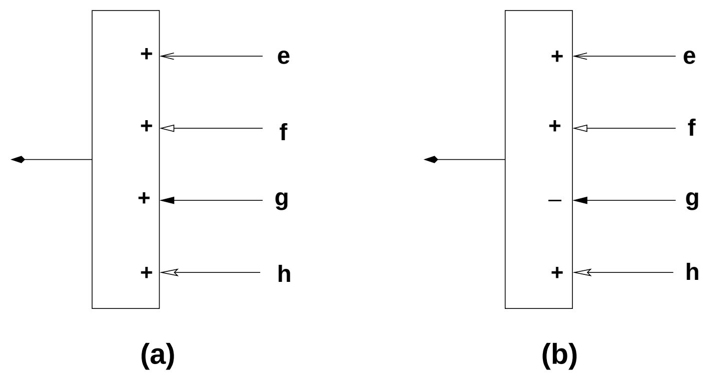
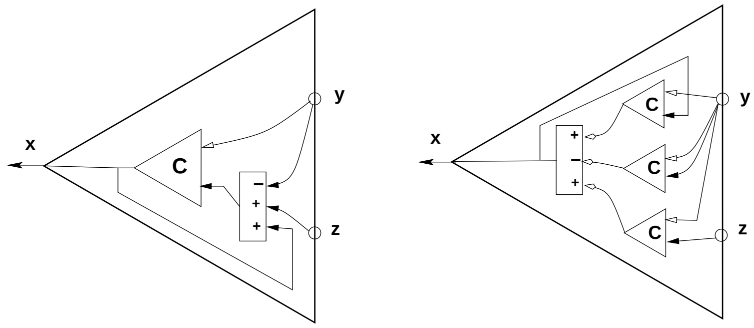


Figure 2.4: Add and Add-Subtract cells

cell always has symmetric inputs though the symmetry of the inputs will not have implications for our intended applications. We can vary the picture a little by combining one or more inputs to an Add cell with a scale cell with $s = -1$. In this way we can do arbitrary addition and subtraction. We show in Figure 2.4(b) an Add-Subtract cell with three additive inputs and one subtractive. The output of this cell is $\mathbf{e} + \mathbf{f} - \mathbf{g} + \mathbf{h}$. In the sequel, we use the notation shown in Figure 2.4 for Add and Add-Subtract cells.

Example 2.1.10. We may use Add, Add-Subtract, and Scaling cells to build new cells from old. In Figure 2.5 we show two characteristic examples. We start with a cell $\mathbf{C}$ which we suppose has two inputs. Assume a differential equation model with *linear* phase space $\mathbb{R}^N$. In Figure 2.5(a) we show a new two-input cell constructed using $\mathbf{C}$ and an Add-Subtract cell. If dynamics on $\mathbf{C}$ is defined by $f : \mathbb{R}^N \times (\mathbb{R}^N)^2 \rightarrow \mathbb{R}^N$, then dynamics on the cell defined in Figure 2.5(a) is given by $F : \mathbb{R}^N \times (\mathbb{R}^N)^2 \rightarrow \mathbb{R}^N$, where $F(x; y, z) = f(x; y, x - y + z)$. This model would also work if the phase space for $\mathbf{C}$ was the N-torus $\mathbb{T}^N$. The same model also works



(a) (b)

Figure 2.5: Building new cells using Add, Add-Subtract cells

for discrete cells provided that the phase space is either $\mathbb{R}^N$ or $\mathbb{T}^N$. However, the model does not work if the phase space for **C** is a general manifold – for example, the two-sphere S^2.

In Figure 2.5(b) we combine outputs rather than inputs. The corresponding vector field for the cell shown in Figure 2.5(b) is given by

$$F(x; y, z) = f(x; y, x) - f(x; y, y) + f(x; y, z).$$

This is valid for continuous dynamics on an arbitrary smooth manifold.

Remarks 2.1.11. (1) For discrete dynamics on a manifold, we can linearly combine outputs provided that $f(x_0; x_1, \ldots, x_r)$ is sufficiently close to x_0, for all $x_0, \ldots, x_r \in M$. For this it suffices to fix a Riemannian metric on M and then do addition and scalar multiplication in the tangent space $T_{x_0}M$ using the exponential map of the metric (that is, $\exp_{x_0} : T_{x_0}M \rightarrow M$ is a local diffeomorphism on a neighborhood of the origin of $T_{x_0}M$). Note that this approach does not work for inputs as there is no reason to assume that linear combinations of $x_0, \ldots, x_r$ need be close to x_0.

Once we have fixed a Riemannian metric on M, we can always linearly combine outputs for the discretization of the differential equation (at least if the time step is sufficiently small).

(2) For continuous dynamics, we interpret Figure 2.5(b) in the following way: we assume the outputs of $\mathbf{C}$ are un-integrated – that is vector fields. We then linearly combine, scale and integrate to get the output. If we go to the discretization, then the previous remark holds (cf. Remarks 2.1.7(2)).

Example 2.1.12 (Choose and Pick cell)**.** We introduce one further construction that will be useful in simplifying the diagrams we use for some of the examples (this gadget is not used in any of the proofs).

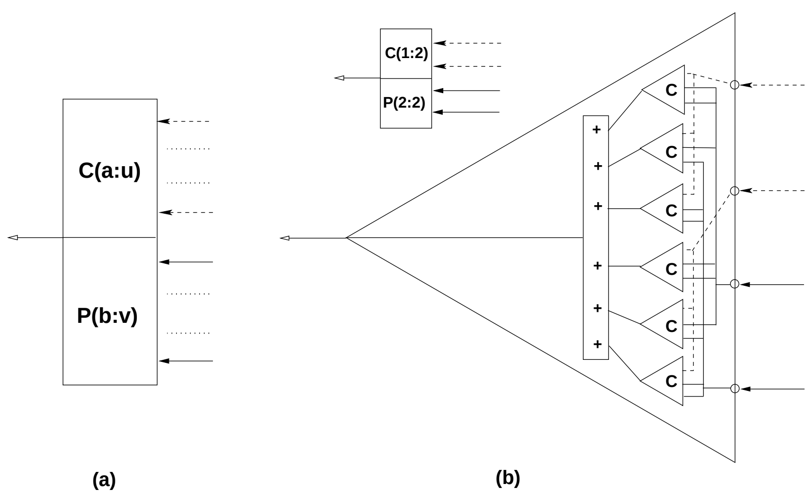


Figure 2.6: Choose and pick cells $\mathbf{C}(a, b : u, v)$, $\mathbf{C}(1, 2 : 2, 2)$

Let $\mathbf{C}$ be a cell which has r inputs all of the same type (we may allow other input types, but they will not effect the construction which only affects inputs of one type). Suppose that continuous dynamics are determined by the vector field $h : M \times M^r \to TM$, where $h(x_0; x_1, \cdots, x_r)$ is symmetric in the variables $x_1, \cdots, x_r$.

Suppose that $a, b, u, v \in \mathbb{N}$ satisfy $u + v = r$ and $a \le u$, $b \le v$. The *choose and pick cell* $\mathbf{C}(a, b : u, v)$ is a new cell built by adding outputs from $\binom{v+b-1}{b}\binom{u}{a}$ class $\mathbf{C}$ cells. The cell $\mathbf{C}(a, b : u, v)$ will have two input types; u inputs of the first type, v inputs of the second type. More precisely, the cell $\mathbf{C}(a, b : u, v)$ has two components, denoted $C(a : u)$ and $P(b : v)$, corresponding to the two input types. If $x_1, \cdots, x_u$ are the inputs to the $C(a : u)$ component and $y_1, \cdots, y_v$ are the inputs to the $P(b : v)$ component, then the output of $\mathbf{C}(a, b : u, v)$ is defined to be

$$\sum_{\substack{1 \le j_1 \le \cdots \le j_b \le v \\ 1 \le i_1 < \cdots < i_a \le u}} h(x_0; y_{j_1}, \cdots, y_{j_b}, x_{i_1}, \cdots, x_{i_a}).$$

The output of $\mathbf{C}(a, b : u, v)$ is symmetric in $x_1, \cdots, x_u$ and $y_1, \cdots, y_v$. We use the symbol for $\mathbf{C}(a, b : u, v)$ shown in Figure 2.6(a). In Figure 2.6(b), we show the connections for the choose and pick cell $\mathbf{C}(1, 2 : 2, 2)$.

2.1.4 Some special classes of coupled cell networks

Throughout this section, $\mathcal{M}$ will denote a fixed network architecture. We have already indicated that we also regard $\mathcal{M}$ as the set of all coupled cell systems with network architecture $\mathcal{M}$. Henceforth we assume that any coupled cell system $\mathcal{S} \in \mathcal{M}$ always has a phase space which is a smooth connected differential manifold.

It is appropriate to single out some special classes of coupled cell networks. The restrictions we impose are on the phase space and connection structure rather than on the type of the system (continuous, discrete, hybrid, etc).

1. $\mathcal{M}(\mathbb{L})$ denotes the set of systems $\mathcal{S} \in \mathcal{M}$ which have linear phase space.

2. $\mathcal{M}(\mathbb{T})$ denotes the set of systems for which the phase space is a compact connected Abelian Lie group – that is, an N-torus for some $N \geq 1$.

3. $\mathcal{M}(\mathbb{G})$ denotes the set of systems for which the phase space is a Lie group.

We extend this notation to consider networks with specific phase spaces. For example, let $\mathcal{M}(\mathbb{R})$ denotes the set of systems $\mathcal{S} \in \mathcal{M}(\mathbb{L})$ which have phase space $\mathbb{R}$ and $\mathcal{M}(\mathrm{SO}(3))$ denote the class of systems $\mathcal{S} \in \mathcal{M}(\mathbb{G})$ which have phase space SO(3). We remark that $\mathcal{M} \supset \mathcal{M}(\mathbb{G}) \supset \mathcal{M}(\mathbb{L}), \mathcal{M}(\mathbb{T})$ and $\mathcal{M}(\mathbb{L}) \supset \mathcal{M}(\mathbb{R})$.

Scalar signalling networks

As we have remarked previously, from the point of view of applications it is unrealistic to assume that in a large network each cell has to have access to complete knowledge of the state of cells from which it receives outputs. (Of course, complete information may be important in small networks of cells with low dimensional phase spaces.) Suppose then that we have an identical cell system comprised of cells of class **C**. Denote the phase space of **C** by M. Let $\xi : M \to \mathbb{F}$ be a smooth function, where $\mathbb{F}$ denotes either the real or complex numbers[3]. If $x(t)$ is a trajectory on M, then $\xi(x(t))$ will be a curve in $\mathbb{F}$. While we could regard ξ as an observable, in our context we prefer to think of ξ a signal. For example, in neuronal dynamics, $\xi(x(t))$ might typically be zero or small except when the neuron spikes.

Definition 2.1.13. An identical coupled cell system $\mathcal{S} \in \mathcal{M}$ is a *scalar signalling system* if there exists a signal $\xi : M \to \mathbb{F}$ defined on the phase space of each cell

[3]More generally, $\mathbb{F}$ could be a finite dimensional vector space

such that the inputs to each cell depend only on the signals from the corresponding output cells. Let $\mathcal{M}(\mathbb{S})$ denote the class of scalar signalling systems $\mathcal{S} \in \mathcal{M}$.

Example 2.1.14. Let $\mathcal{S}$ be a scalar signalling system with the network of Figure 2.2(ii). Let M be the phase space of a cell and $\xi : M \to \mathbb{F}$ be a signal. Set $\xi \circ x = \hat{x}$. With these notational conventions and a continuous differential equation model, the differential equations for the system will be

$$\begin{aligned} x_1' &= f(x_1; \hat{x}_2, \hat{x}_1, \hat{x}_3), \\ x_2' &= f(x_2; \hat{x}_1, \hat{x}_3, \hat{x}_3), \\ x_3' &= f(x_3; \hat{x}_1, \hat{x}_2, \hat{x}_2). \end{aligned}$$

Note that the internal variables are *not* changed.

Remarks 2.1.15. (1) A key feature of scalar signalling systems is that we can linearly combine inputs even though the phase space may be nonlinear. In particular, the configuration shown in Figure 2.5(a) is valid for both continuous *and* discrete scalar signalling systems irrespective of the phase space.
(2) The generalization of the definition of scalar signalling systems to networks with multiple cell classes is completely straightforward. This extension is significant as the main application we have in mind for scalar signalling systems is the coupling together of small networks (not scalar signalling) into large networks with scalar signalling between the small networks. In this context, it may well be appropriate to assume that there are no self loops around the small networks (see Remark 2.1.3(2)) even though the cells within the small networks may have self loops. As a simple illustration, an audio amplifier internally may have several

negative feedback loops but feeding an output of the complete audio system back into the audio source (say, a microphone) is generally not advisable. Similar observations apply in control theory.

(3) Finally, we will abuse notation and refer, for example, to a coupled cell network $\mathcal{M}(\mathbb{L})$. By this we mean that the network architecture is $\mathcal{M}$ but we will always restrict to systems with linear phase space. Similar remarks hold for the other network classes we have defined. The reason we do this is that we shall be defining various relations and orders on networks and some of these definitions only apply if we assume extra structure on the connections or the phase space.

(4) Scalar signals can be used to stabilize a system (see Example 2.1.16). Example 2.1.18 gives a system which cannot be stabilized by any scalar signal.

Example 2.1.16. Let $\mathcal{M}$ be the coupled cell network shown in Figure 4.1. Assume that the phase space of each cell is two dimensional. Let the inputs be scalar so that the coupled cell system is given by

$$\mathbf{x}' = F(\mathbf{x}; \xi(\mathbf{x}), \xi(\mathbf{y})), \quad \mathbf{y}' = F(\mathbf{y}; \xi(\mathbf{x}), \xi(\mathbf{x})), \tag{2.1.3}$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$, $F = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$, $\xi : \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ signal. Let $\xi(0,0) = 0, F((0,0);0,0) = (0,0)$. Then $\mathbf{O} = (0,0,0,0,0,0) \in \mathbb{R}^6$ is an equilibria of (2.1.3).

For $x_i \in \mathbb{R}$, $i = 0, \cdots, 3$, $F((x_0, x_1); x_2, x_3) \in \mathbb{R}^2$. For $i = 1, 2$, let $\frac{\partial f_i}{\partial x_0}(\mathbf{O}) = a_i$, $\frac{\partial f_i}{\partial x_1}(\mathbf{O}) = b_i$, $\frac{\partial f_i}{\partial x_2}(\mathbf{O}) = c_i$, $\frac{\partial f_i}{\partial x_3}(\mathbf{O}) = d_i$. Also for $x_0, x_1 \in \mathbb{R}^2$, $\xi(x_0, x_1) \in \mathbb{R}$. Let $\frac{\partial \xi}{\partial x_0}((0,0)) = u$, $\frac{\partial \xi}{\partial x_1}((0,0)) = v$. Then the linearization of (2.1.3) at $\mathbf{O}$ is

$\mathbf{J} = \begin{pmatrix} A+B & C \\ B+C & A \end{pmatrix}$, where $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $B = \begin{pmatrix} c_1 u & c_1 v \\ c_2 u & c_2 v \end{pmatrix}$, $C = \begin{pmatrix} d_1 u & d_1 v \\ d_2 u & d_2 v \end{pmatrix}$.

The eigenvalues of $\mathbf{J}$ are the eigenvalues of $A+B+C$ and $A-C$. The necessary and sufficient conditions for $\mathbf{O}$ to be an asymptotically stable equilibria are

(T) $\text{trace}(A+B+C) < 0$, $\text{trace}(A-C) < 0$,

(D) $\det(A+B+C) > 0, \det(A-C) > 0$.

Let us simplify these conditions. Note that $\text{trace}(A+B+C) = (a_1+b_2) + (c_1+d_1)u + (c_2+d_2)v < 0$, and $\text{trace}(A-C) = (a_1+b_2) - (d_1 u + d_2 v) < 0$. Also, note that $\det(A+B+C) > 0$ simplifies to $a_1 b_2 - a_2 b_1 > u(b_1 c_2 - b_2 c_1 + b_1 d_2 - b_2 d_1) - v(a_1 c_2 - a_2 c_1 + a_1 d_2 - a_2 d_1)$, and $\det(A-C) > 0$ simplifies to $a_1 b_2 - a_2 b_1 > -u(b_1 d_2 - b_2 d_1) + v(a_1 d_2 - a_2 d_1)$. For notational convenience, let $\alpha = b_1 c_2 - b_2 c_1$, $\beta = b_1 d_2 - b_2 d_1$, $\gamma = a_1 c_2 - a_2 c_1$, $\delta = a_1 d_2 - a_2 d_1$, $p = \det(A)$, $q = \text{trace}(A)$.

Theorem 2.1.17. *The necessary and sufficient conditions for* $\mathbf{O}$ *to be an asymptotically stable equilibria for (2.1.3) are*

$$\begin{aligned} (T): q &< \min\{-(c_1+d_1)u - (c_2+d_2)v, d_1 u + d_2 v\}, \\ (D): p &> \max\{(\alpha+\beta)u - (\gamma+\delta)v, -\beta u + \delta v\}. \end{aligned}$$

Let $p < 0$, $c_1 d_2 - c_2 d_1 > 0$ ($\Rightarrow \alpha\delta - \beta\gamma = -p(c_1 d_2 - c_2 d_1) > 0$), $c_2 + d_2 > 0$, $d_2 > 0$, $c_1 + d_1 > 0$, $d_1 > 0$, $c + d > 0$, $d > 0$.

Case ($q > 0$): Condition (T) gives $v < -\frac{c_1+d_1}{c_2+d_2}u - \frac{q}{c_2+d_2}$, $v > -\frac{d_1}{d_2}u + \frac{q}{d_2}$, and these conditions define a region in the plane contained in the second open quadrant.

Condition (D) gives $v > \frac{\alpha+\beta}{\gamma+\delta}u - \frac{p}{\gamma+\delta}$, $v < \frac{\beta}{\delta}u + \frac{p}{\delta}$, and these conditions define a region in the plane contained in the third open quadrant. Therefore, there are no values of (u, v) for which (2.1.3) can be stabilized.

Case $(q < 0)$: The point of intersection of the lines given by equalities in (T) in the (u, v)-plane is $P = \left(-\frac{q(c_2+2d_2)}{c_1d_2-c_2d_1}, \frac{q(c_1+2d_1)}{c_1d_2-c_2d_1}\right)$. The point of intersection of the lines given by equalities in (D) in the (u, v)-plane is $Q = \left(\frac{p(\gamma+2\delta)}{\alpha\delta-\beta\gamma}, \frac{p(\alpha+2\beta)}{\alpha\delta-\beta\gamma}\right)$. The point P lies in fourth quadrant, and Q lies in third quadrant. It can be easily seen that (2.1.3) can be stabilized if and only if Q lies in the feasible region of (T). That is,

$$\frac{p\left(d_2(\alpha+2\beta)+d_1(\gamma+2\delta)\right)}{\alpha\delta-\beta\gamma} > q.$$

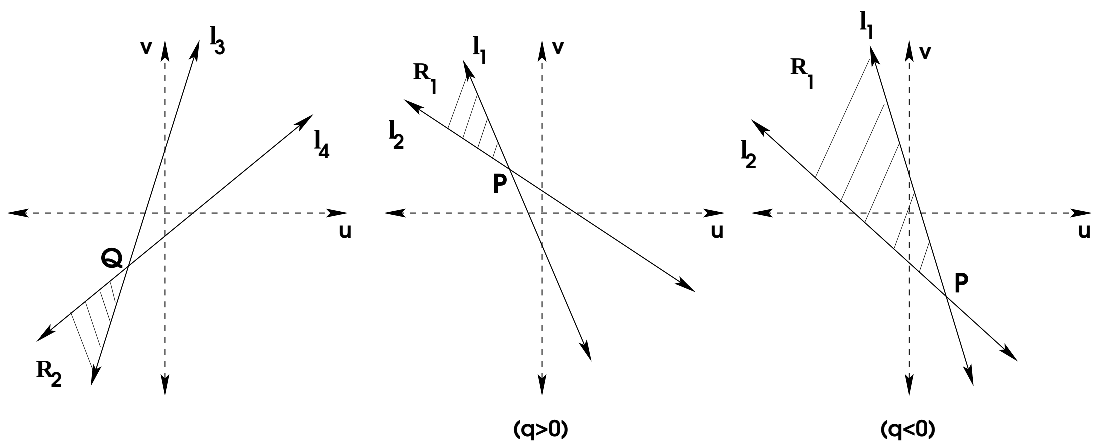


Figure 2.7: The lines $\mathbf{l_1}$: $v = -\frac{c_1+d_1}{c_2+d_2}u - \frac{q}{c_2+d_2}$, $\mathbf{l_2}$: $v = -\frac{d_1}{d_2}u + \frac{q}{d_2}$, $\mathbf{l_3}$: $v = \frac{\alpha+\beta}{\gamma+\delta}u - \frac{p}{\gamma+\delta}$, $\mathbf{l_4}$: $v = \frac{\beta}{\delta}u + \frac{p}{\delta}$, $\mathbf{R_1}$: feasibility region of (T), $\mathbf{R_2}$: feasibility region of (D)

Example 2.1.18. Let $\mathcal{M}$ be the bidirectional ring of two cells. Assume that the phase space of each cell is two dimensional. Let the inputs be scalar so that the coupled cell system is given by

$$\mathbf{x}' = F(\mathbf{x}; \xi(\mathbf{y})), \;\; \mathbf{y}' = F(\mathbf{y}; \xi(\mathbf{x})), \tag{2.1.4}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $F = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$, $\xi : \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ signal. Let $\xi(0,0) = 0, F((0,0); 0) = (0,0)$. Then $\mathbf{O} = (0,0,0,0) \in \mathbb{R}^4$ is an equilibria of (2.1.4).
For $x_i \in \mathbb{R}$, $i = 0, \cdots, 2$, $F((x_0, x_1); x_2) \in \mathbb{R}^2$. For $i = 1, 2$, let $\frac{\partial f_i}{\partial x_0}(\mathbf{O}) = a_i$, $\frac{\partial f_i}{\partial x_1}(\mathbf{O}) = b_i$, $\frac{\partial f_i}{\partial x_2}(\mathbf{O}) = c_i$. Also for $x_0, x_1 \in \mathbb{R}^2$, $\xi(x_0, x_1) \in \mathbb{R}$. Let $\frac{\partial \xi}{\partial x_0}((0,0)) = u$, $\frac{\partial \xi}{\partial x_1}((0,0)) = v$. Then the linearization of (2.1.4) at $\mathbf{O}$ is $\mathbf{J} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$, where $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $B = \begin{pmatrix} c_1 u & c_1 v \\ c_2 u & c_2 v \end{pmatrix}$. The eigenvalues of $\mathbf{J}$ are the eigenvalues of $A + B$ and $A - B$. The necessary and sufficient conditions for $\mathbf{O}$ to be an asymptotically stable equilibria are

(T) $\mathrm{trace}(A + B) < 0$, $\mathrm{trace}(A - B) < 0$,

(D) $\det(A + B) > 0, \det(A - B) > 0$.

Let us simplify these conditions. Condition (T) implies $\mathrm{trace}(A) < 0$. Also note that (D) implies $\det(A) > |au - cv| > 0$. Therefore, $\mathbf{O}$ cannot be made an asymptotically stable equilibria unless $\mathrm{trace}(A) < 0$ and $\det(A) > 0$.

2.2 Generalities on graphs

Every coupled cell network determines a directed graph. In this section, we briefly review those aspects of graph theory that relate to coupled cell networks. A directed graph is a set of vertices or cells and directed edges or connections. Because of our interest in networks of coupled dynamical systems, we often use term ‘cell’ rather than ‘vertex’ and ‘connection’ rather than ‘edge’. Viewed in this way, each cell admits inputs (from other cells) and has outputs (to other cells). Figure 2.8(a) shows an example of a directed graph represented in the way customary in graph theory (or in the works on coupled cell systems by Stewart et al. [49, 25]). Each node consists of a cell with two identical inputs (there is only one edge type). Here, we prefer to represent the graph as shown in Figure 2.8(b). The reason we adopt

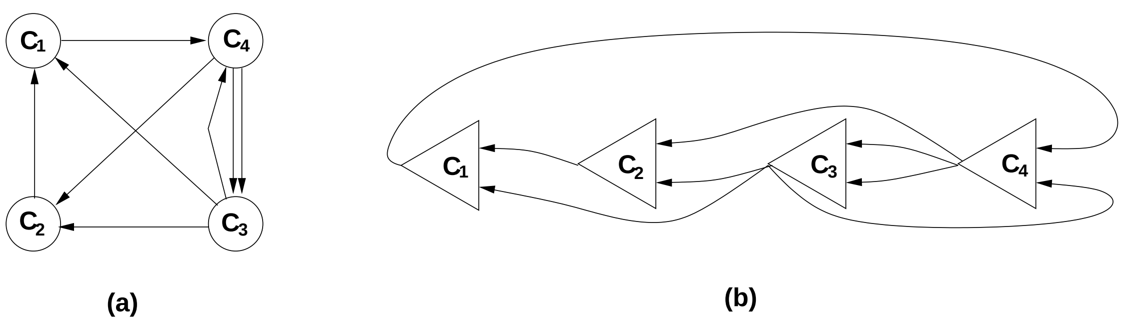


Figure 2.8: Two representations of the same directed graph

this ‘linear systems’ representation is because it fits better with our definition of inflation (see Chapter 6). Indeed, the assumption of a linear (horizontal) structure allows us to represent inflation as occurring in the vertical direction.

Associated to every directed graph, we may define the *adjacency matrix* of the graph. We recall the definition. Suppose that $\mathcal{G}$ is a directed graph with vertex set $V = \{v_1, \cdots, v_k\}$. Define the $k \times k$ adjacency matrix $A = [a_{ij}]$ of $\mathcal{G}$ by $a_{ij} = p$ if there are exactly p edges from v_i to v_j. If there are no edges from v_i to v_j, then $p = 0$. Note that this definition takes no account of edge type. For $i \in \mathbf{k}$, the *indegree* of the vertex v_j is the j^{th} column sum of A and is the total number of inputs to the vertex v_j. Similarly, for $i \in \mathbf{k}$, the *outdegree* of the vertex v_i is the i^{th} row sum of A and is the total number of outputs from the vertex v_i. A *path* in the graph $\mathcal{G}$ is a sequence of vertices and directed edges such that from each vertex there is an edge to the next vertex in the sequence. A *chain* is a closed path; that is, the two terminal vertices of the sequence are the same. A *self loop* (loop) is an edge that connects a vertex to itself. $\mathcal{G}$ is said to be *strongly connected* if there is a path from each vertex to every other vertex. A necessary condition for a graph to be strongly connected is that each vertex has an input and an output. In terms of the adjacency matrix, this means that each row and each column of the adjacency matrix of $\mathcal{G}$ must have a nonzero entry.

Suppose that $\mathcal{G}$ is a graph with k vertices and ℓ edge types. Associated to each edge type $r \in \boldsymbol{\ell}$, we may also define an adjacency matrix $A_r = [a_{ij}^r]$ of type r by $a_{ij}^r = p$ if there are exactly p edges of type r from v_i to v_j. If there are no edges of type r from v_i to v_j, then $p = 0$. We refer to $A_1, \cdots, A_\ell$ as the *edge type adjacency matrices* of $\mathcal{G}$. We remark that $A = A_1 + \cdots + A_\ell$. We refer to [14] for detailed theory on graphs.

A square non-negative matrix $A = [a_{ij}]$ of order k is said to be *irreducible* if for each $1 \leq i, j \leq k$, there exists $n \in \mathbb{N}$ such that $(A^n)_{ij} > 0$. If A is the adjacency matrix of graph $\mathcal{G}$, then A is irreducible if and only if the graph $\mathcal{G}$ is strongly connected. The Perron Frobenius theorem [35] for irreducible matrices states that if the adjacency matrix A with period[4] p and spectral radius $\rho(A) = r$ is irreducible then r is a positive simple eigenvalue. Moreover, if $p > 1$ then there

[4]For each $i \leq i \leq k$, the greatest common divisor p_i of the positive integers m such that $(A^m)_{ii} > 0$. For an irreducible matrix $p_i = p$, for all i.

is a permutation matrix P such that PAP^{-1} takes the following form

$$\begin{pmatrix} 0 & A_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & 0 & \cdots & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & A_{p-1} \\ A_p & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where the zeros on the main diagonal correspond to square zero matrices.

2.3 Synchrony class

We recall the key aspects of the definition of a *synchrony subspace* of $\mathcal{M}$ (we refer to [4] or [25] for a more extended description of synchrony subspaces and note that what we refer to as a synchrony subspace is called a 'polydiagonal' subspace in [25]). Let $\mathcal{S} = \{S_1, \cdots, S_p\}$ be a partition of $\{C_1, \cdots, C_k\}$. Let $s(j) \geq 1$ be the number of cells in S_j, $1 \leq j \leq p$. To avoid trivialities, we always assume $p < k$ and so at least one $s(j) > 1$. Let $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_k)$ denote the state of the cells in $\mathcal{M}$ – since the cells are assumed identical, they all have the same phase space. Group the components of $\mathbf{x}$ according to the partition S and write $\mathbf{x} = (\mathbf{x}^1, \cdots, \mathbf{x}^p)$, where $\mathbf{x}^j = (\mathbf{x}_{j1}, \cdots, \mathbf{x}_{js(j)})$ denotes the state of the $s(j)$ cells in S_j, $1 \leq j \leq p$. Define

$$\mathcal{X}_j = \{\mathbf{x}^j \mid \mathbf{x}_{j1} = \cdots = \mathbf{x}_{js(j)}\},\ 1 \leq j \leq p,$$

and

$$\mathcal{X}(S) = \{\mathbf{x} = (\mathbf{x}^1, \cdots, \mathbf{x}^p) \mid \mathbf{x}^j \in \mathcal{X}_j, 1 \leq j \leq p\}.$$

The partition S is said to be a *synchrony class* if for every realization of the network architecture $\mathcal{M}$ as a coupled cell system, $\mathcal{X}(S)$ is an invariant subspace for the dynamics of that system. If S is a synchrony class then we say that $\mathcal{X}(S)$ is a *synchrony subspace* for the system. In dynamical terms, suppose $\mathbf{x}_0$ is the initial state of the system (at time $t = 0$) and $\mathbf{x}_0 \in \mathcal{X}(S)$. If $\mathcal{X}(S)$ is a synchrony subspace, then $\mathbf{x}(t) \in \mathcal{X}(S)$ for all $t \geq 0$ (t may be continuous or discrete). As a result, the cells in each of the sets S_j will be synchronized (have the same state) for all $t > 0$.

Following the notation conventions of [4], we denote the synchrony subspace (or class) by $\{S_1 \| \cdots \| S_p\}$ (we typically omit partition elements S_i if S_i consists of a single cell). As a simple example, suppose $S_1 = \{C_{i_1}, \cdots, C_{i_s}\}$. Either of the notations $\{S_1\}$, $\{C_{i_1}, \cdots, C_{i_s}\}$ would indicate a single group of synchronized cells. On occasions, it is sometimes convenient to identify synchrony classes by means of state variables rather than cell labels. Thus, if cell C_i has state variable $\mathbf{x}_i$, $\{\mathbf{x}_{i_1}, \cdots, \mathbf{x}_{i_s}\}$ would be an alternative notation for the synchrony class $\{C_{i_1}, \cdots, C_{i_s}\}$. The definition of synchrony subspace (or class) is easily generalized to networks with non-identical cells.

Remark 2.3.1. For networks with identical cells, it is proved in [26, Theorem 4.3], [23, Theorem 6.5] that the following are equivalent.

1. S is a synchrony space.

2. for each input type ℓ, the number of inputs of type ℓ from cells in S_i to each cell in S_j is the same, for $i, j \in \ell$.

Example 2.3.2. Consider the following coupled cell system with network architecture $\mathcal{M}$ (the underlying graph has four cells and is fully connected).

$$\begin{aligned} \theta_1' &= f(\theta_1;\theta_2,\theta_4,\theta_3), & \theta_2' &= f(\theta_2;\theta_1,\theta_3,\theta_4), \\ \theta_3' &= f(\theta_3;\theta_4,\theta_2,\theta_1), & \theta_4' &= f(\theta_4;\theta_3,\theta_1,\theta_2), \end{aligned}$$

where $f : \mathbb{T} \times \mathbb{T}^3 \to \mathbb{T}$ (the system can be a discrete dynamical system as well where instead of differential equations, we have regular updates of the state of each cell or oscillator by the map f). If we assume that all the three inputs to each cell are asymmetric then the synchrony subspaces of $\mathcal{M}$ are $\{\theta_1 = \theta_2 = \theta_3 = \theta_4\}$, $\{\theta_1 = \theta_2, \theta_3 = \theta_4\}$, $\{\theta_1 = \theta_4, \theta_2 = \theta_3\}$, and $\{\theta_1 = \theta_3, \theta_2 = \theta_4\}$.

A framework for coupled cell systems has been presented by Golubitsky and coworkers [26, 24] that permits a classification of synchrony spaces in terms of the concept of a 'balanced equivalence relation', which depends solely on the network architecture. Stewart [44] and Kamei [33] proved that the set of all balanced equivalence relations on a network forms a lattice (a partially ordered set in which any two elements have a meet and a join, the partial order is defined by refinement).

Chapter 3

Dynamical Equivalence

In Chapters 3, 4, and 5, we develop various notions of equivalence for coupled cell networks. We follow the approach of Aguiar *et al.* [4] and concentrate on dynamics rather than adopt the more abstract viewpoint of Dias and Stewart [18] which is based on groupoid formalism and restricted to systems with linear phase space.

Suppose that $\mathcal{F}$ and $\mathcal{G}$ are coupled cell systems which both have n cells (which need not be identical). Label the cells of $\mathcal{F}$ and $\mathcal{G}$ as $\{\mathbf{C}_1, \ldots, \mathbf{C}_n\}$ and $\{\mathbf{D}_1, \ldots, \mathbf{D}_n\}$ respectively. Assume that both systems have the same type of dynamics: either both discrete or both continuous. For ease of exposition we henceforth assume continuous dynamics modelled by ordinary differential equations but emphasize that most of what we say applies equally well to discrete dynamics. We indicate in the remarks when it does not.

The coupled cell systems $\mathcal{F}$, $\mathcal{G}$ have *identical dynamics* if

1. The cells $\mathbf{C}_i$, $\mathbf{D}_i$ have the same phase space, $i \in \mathbf{n}$.

2. The time evolution of both systems is identical.

Remarks 3.0.3. (1) If $\mathcal{F}$ and $\mathcal{G}$ have identical dynamics it does not follow that $\mathcal{F}$ and $\mathcal{G}$ have the same network architecture or that corresponding cells have the same number of inputs.
(2) Note that the definition of identical dynamics requires no restrictions on the type of phase space or connections.

We define a partial ordering on coupled cell networks, and an associated equivalence relation.

Definition 3.0.4 ([4])**.** Let $\mathcal{N}, \mathcal{M}$ be coupled cell networks both with n cells.

(a) The network $\mathcal{N}$ is *dominated* by $\mathcal{M}$, denoted $\mathcal{N} \prec \mathcal{M}$, if given an ordering of the cells in $\mathcal{N}$, we can choose an ordering of the cells of $\mathcal{M}$ such that given any system $\mathcal{F} \in \mathcal{N}$, there exists a system $\mathcal{F}^\star \in \mathcal{M}$ such that $\mathcal{F}$ and $\mathcal{F}^\star$ have identical dynamics.

(b) We say $\mathcal{N}$ and $\mathcal{M}$ are *equivalent*, denoted $\mathcal{N} \sim \mathcal{M}$, if we can order the cells in $\mathcal{M}$ and $\mathcal{N}$ so that $\mathcal{N} \prec \mathcal{M}$ and $\mathcal{M} \prec \mathcal{N}$.

Remarks 3.0.5. (1) In Definition 3.0.4(b) it is not necessary to assume that the orderings of cells for which $\mathcal{N} \prec \mathcal{M}$ and $\mathcal{M} \prec \mathcal{N}$ are the same. Indeed, if they are not it is easy to see that we obtain a non-trivial permutation of the ordering of the first ordering of $\mathcal{M}$ relative to which $\mathcal{M} \sim \mathcal{M}$. Since we are assuming finitely many cells in $\mathcal{M}$, the order of the permutation is finite and from this we easily deduce that we can choose an ordering of the cells of $\mathcal{M}$ and $\mathcal{N}$ for which we have $\mathcal{N} \prec \mathcal{M}$ and $\mathcal{M} \prec \mathcal{N}$.
(2) If we restrict attention to systems with linear phase space, we can also define linear equivalence [18]. We write $\mathcal{M}(\mathbb{L}) \prec_L \mathcal{N}(\mathbb{L})$ if given $\mathcal{F} \in \mathcal{M}(\mathbb{L})$ modelled by a

linear differential equation, there exists $\mathcal{F}^\star \in \mathcal{N}(\mathbb{L})$ modelled by a linear differential equation, such that $\mathcal{F}$ and $\mathcal{F}^\star$ have identical dynamics. We say the networks $\mathcal{N}$ and $\mathcal{M}$ are *linearly equivalent*, denoted by $\mathcal{M}(\mathbb{L}) \sim_L \mathcal{N}(\mathbb{L})$, if $\mathcal{M}(\mathbb{L}) \prec_L \mathcal{N}(\mathbb{L})$ and $\mathcal{N}(\mathbb{L}) \prec_L \mathcal{M}(\mathbb{L})$. (We remind the reader of the abuse of notation indicated in Remarks 2.1.15(3).)

For the remainder of the chapter we assume identical cell networks. For $n, m \in \mathbb{N}$, let $M(n, m; \mathbb{K})$ denote the space of $n \times m$-matrices with entries in $\mathbb{K}$ where $\mathbb{K}$ will be either $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{Z}^+$. In case $m = n$, set $M(n, m; \mathbb{K}) = M(n; \mathbb{K})$.

Let $\mathcal{M}$ be an r-input n cell network with adjacency matrices $M_0 = I, M_1, \ldots, M_p$. Let $\mathbf{A}(\mathcal{M})$ denote the vector subspace of $M(n; \mathbb{Q})$ spanned by $M_0, \ldots, M_p$ and $\mathbf{A}(\mathcal{M}; \mathbb{Z}^+)$ denote the set of all non-negative integer combinations of $M_0, M_1, \cdots, M_p$. We emphasize that to define the adjacency matrices we fix an ordering of the cells. In particular, the space $\mathbf{A}(\mathcal{M})$ will depend on the choice of ordering (but not on the ordering of input types). We recall the main result of [18] adapted to our context and notational conventions.

Theorem 3.0.6 ([18])**.** *Let $\mathcal{M}, \mathcal{N}$ be coupled cell networks both with n cells. The following conditions are equivalent:*

1. $\mathcal{M}(\mathbb{L}) \prec \mathcal{N}(\mathbb{L})$. *(Phase space linear.)*

2. $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$.

3. $\mathcal{M}(\mathbb{L}) \prec_L \mathcal{N}(\mathbb{L})$.

4. $\mathcal{M}(\mathbb{R}) \prec_L \mathcal{N}(\mathbb{R})$. *(Phase space $\mathbb{R}$).*

In particular, $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ *iff* $\mathcal{M}(\mathbb{L}) \sim \mathcal{N}(\mathbb{L})$.

Remarks 3.0.7. (1) In all statements it is assumed that there is a given ordering of cells in $\mathcal{M}$ and that we can choose an ordering of cells in $\mathcal{N}$ for which the corresponding statement holds. (2) Theorem 3.0.6 does not apply to systems with non-linear phase space. Indeed, (3,4) have no meaning when the phase space is non-linear and the methods used in [18] do not apply to prove the equivalence of (1) and (2) for the case of non-linear phase spaces. (3) Theorem 3.0.6 applies when the network is governed by discrete dynamics and phase spaces are linear.

As a corollary of Theorem 3.0.6 we have

Theorem 3.0.8. *Let $\mathcal{M}, \mathcal{N}$ be coupled cell networks both with n cells. A necessary condition for $\mathcal{M} \sim \mathcal{N}$ is $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$.*

Proof. If $\mathcal{M} \sim \mathcal{N}$ then $\mathcal{M}(\mathbb{L}) \sim \mathcal{N}(\mathbb{L})$ and so the result follows from Theorem 3.0.6. $\square$

Remark 3.0.9. It is easy to verify Theorem 3.0.8 directly. Specifically, the difficult part of the proof of Theorem 3.0.6 involves the verification that $\mathcal{M}(\mathbb{R}) \prec_L \mathcal{N}(\mathbb{R})$ implies $\mathcal{M}(\mathbb{L}) \prec \mathcal{N}(\mathbb{L})$. This implication is not needed for the proof of Theorem 3.0.8.

In the following two Chapters 4 and 5, we will define notions of input and output equivalence for networks. We state and prove necessary and sufficient conditions for dynamical equivalence of two coupled cell networks. We also give examples to illustrate the results.

Chapter 4

Dynamical Equivalence : Networks with Asymmetric Inputs

In this chapter, we will concentrate on networks of coupled dynamical systems with asymmetric inputs. By asymmetric inputs, we mean that all the inputs to each cell are of different type. We will develop the concept of input and output dominance/equivalence for such networks.

4.1 Input equivalence

Input equivalence is applicable to coupled cell systems with linear phase space and to scalar signalling networks (Definition 2.1.13). The basic idea is that a network $\mathcal{N}$ 'input dominates' a network $\mathcal{M}$ if given a system $\mathcal{F} \in \mathcal{M}(\mathbb{L})$, we can find a system $\mathcal{G} \in \mathcal{N}(\mathbb{L})$ which has identical dynamics to $\mathcal{F}$ and is such that each cell in $\mathcal{G}$ is built from one cell in $\mathcal{F}$ together with a number of Add-Subtract and Scaling cells (Section 2.1.3) acting on the inputs (see Figure 2.5(a)). We start with a simple example to illustrate the ideas.

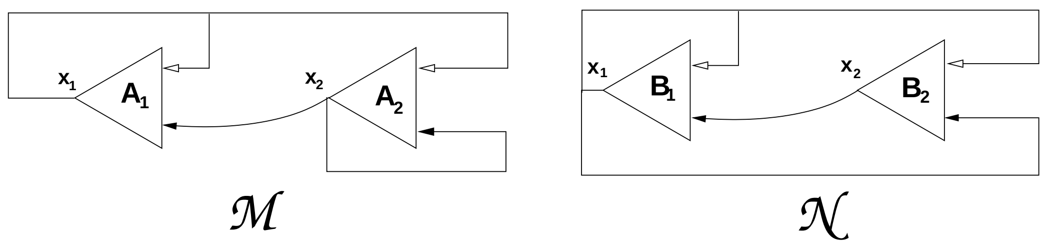


Figure 4.1: $\mathcal{M}$ input dominated by $\mathcal{N}$.

Example 4.1.1 ([4])**.** Referring to Figure 4.1, suppose that $\mathcal{F} \in \mathcal{M}(\mathbb{L})$ is modelled by

$$\begin{aligned} x_1' &= f(x_1; x_1, x_2), \\ x_2' &= f(x_2; x_1, x_2), \end{aligned}$$

where $f = f_{\mathcal{F}} : V \times V^2 \to V$ is a C^1 function on the vector space V. If we define the C^1 model $g = g_{\mathcal{G}}$ for $\mathcal{G} \in \mathcal{N}(\mathbb{L})$ by $g(x_0; x_1, x_2) = f(x_0; x_1, x_0 - x_1 + x_2)$, then

$$\begin{aligned} x_1' &= g(x_1; x_1, x_2) = f(x_1; x_1, x_2), \\ x_2' &= g(x_2; x_1, x_1) = f(x_2; x_1, x_2). \end{aligned}$$

Hence we can realize the dynamics of the first system $\mathcal{F}$ using the second system

$\mathcal{G}$ which has a different architecture. Indeed, we can build the second network using Add-Subtract cells together with the cell used in $\mathcal{F}$. See Figure 4.2 (and also Figure 2.5(a)). Observe that the system shown in Figure 4.2 can be transformed

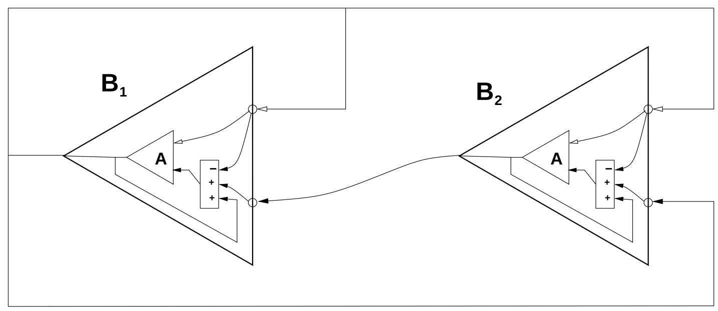


Figure 4.2: The system $\mathcal{G}$ built using the cell from $\mathcal{F}$ and an Add-Subtract cell.

back into the system $\mathcal{F}$ by removing the outer triangles defining cells of class **B** and then cancelling inputs using linearity. We say that the network $\mathcal{M}$ is *input dominated* by $\mathcal{N}$. Finally, the arguments above apply to scalar signaling networks (Definition 2.1.13) – regard $f : V \times \mathbb{R}^2 \to V$ and define $g(x_0; \hat{x}_1, \hat{x}_2) = f(x_0; \hat{x}_1, \hat{x}_0 - \hat{x}_1 + \hat{x}_2)$.

We now extend the previous example and define the concepts of input domination and equivalence for general networks with asymmetric inputs. Conceptually, the idea is quite simple: one network is input dominated by another if the dynamics of any system in the first network can be realized by a system in the second network whose cells are constructed from those in the first network by linearly combining inputs.

Definition 4.1.2. Let V be a vector space, $n \geq 1$, and $r, s \in \mathbb{N}$. Suppose that

$f : V \times V^r \to V$, $g : V \times V^s \to V$ are smooth maps, $\mathfrak{m} = [\mathfrak{m}^1, \ldots, \mathfrak{m}^n] \in M(r, n; \mathbb{Z})$, $\mathfrak{n} = [\mathfrak{n}^1, \ldots, \mathfrak{n}^n] \in M(s, n; \mathbb{Z})$ are connection matrices [1] and $L \in M(r, s+1; \mathbb{Q})$. We say f is $(L, \mathfrak{m}, \mathfrak{n})$-*input dominated by* g, written $f <^{\imath}_{(L,\mathfrak{m},\mathfrak{n})} g$, if

1. For all $(x_0, x_1, \cdots, x_s) \in V \times V^s$, we have

$$g(x_0; x_1, \cdots, x_s) = f(x_0; L(x_0, \cdots, x_s)).$$

2. For $j \in \mathbf{n}$, we have $g(x_j; x_{\mathfrak{n}^j_1}, \cdots, x_{\mathfrak{n}^j_s}) = f(x_j; x_{\mathfrak{m}^j_1}, \cdots, x_{\mathfrak{m}^j_r})$.

Remarks 4.1.3. (1) If f is $(L, \mathfrak{m}, \mathfrak{n})$-input dominated by g then

$$L(x_j, x_{\mathfrak{n}^j_1}, \cdots, x_{\mathfrak{n}^j_s}) = (x_{\mathfrak{m}^j_1}, \cdots, x_{\mathfrak{m}^j_r}),\ j \in \mathbf{n},$$

and so a *necessary* condition for input domination is $\mathfrak{m}^j \subseteq \{j\} \cup \mathfrak{n}^j$, $j \in \mathbf{n}$. That is, $\{\mathfrak{m}^j_1, \ldots, \mathfrak{m}^j_r\} \subset \{j, \mathfrak{n}^j_1, \ldots, \mathfrak{n}^j_s\}$, $j \in \mathbf{n}$. (2) f is *input dominated* by g if we can find L so that $f <^{\imath}_{(L,\mathfrak{m},\mathfrak{n})} g$.
(3) If we can choose $L \in M(r, s+1; \mathbb{Z})$ so that $f <^{\imath}_{(L,\mathfrak{m},\mathfrak{n})} g$, we write $f <^{\imath,\mathbb{Z}}_{(L,\mathfrak{m},\mathfrak{n})} g$.

Definition 4.1.4. Let $\mathcal{M}$ and $\mathcal{N}$ be coupled identical cell networks with asymmetric inputs such that

(a) $n(\mathcal{M}) = n(\mathcal{N}) = n$.

(b) Cells in $\mathcal{M}$ have r inputs, cells in $\mathcal{N}$ have s inputs.

(c) If we fix an ordering $\mathbf{C}_1, \ldots, \mathbf{C}_n$ of the cells in $\mathcal{N}$, then the associated connection matrix is $\mathfrak{n} = [\mathfrak{n}^1, \ldots, \mathfrak{n}^n]$.

[1] see Section 2.1.1.

We say $\mathcal{M}$ is *input dominated* by $\mathcal{N}$, denoted $\mathcal{M} \prec_I \mathcal{N}$, if there exist $L \in M(r, s+1; \mathbb{Q})$ and an ordering of the cells of $\mathcal{M}$, with associated connection matrix $\mathfrak{m}$, such that for every $\mathcal{F} \in \mathcal{M}(\mathbb{L})$ there exists $\mathcal{G} \in \mathcal{N}(\mathbb{L})$ for which $f_{\mathcal{F}} <^{\imath}_{(L,\mathfrak{m},\mathfrak{n})} g_{\mathcal{G}}$. If $\mathcal{N} \prec_I \mathcal{M}$ and $\mathcal{M} \prec_I \mathcal{N}$, we say $\mathcal{M}$ and $\mathcal{N}$ are *input equivalent* and write $\mathcal{M} \sim_I \mathcal{N}$.

Remarks 4.1.5. (1) We write $\mathcal{M} \prec_{I,\mathbb{Z}} \mathcal{N}$ if $\mathcal{M} \prec_I \mathcal{N}$ and we can require the map L of the definition to lie in $M(r, s+1; \mathbb{Z})$. We similarly define $\mathcal{M} \sim_{I,\mathbb{Z}} \mathcal{N}$. In Example 4.1.1 we have $\mathcal{M} \prec_{I,\mathbb{Z}} \mathcal{N}$ (indeed, $\mathcal{M} \sim_{I,\mathbb{Z}} \mathcal{N}$).
(2) Input equivalence and domination may also be defined for networks with symmetric inputs, discussed in the next chapter.

Lemma 4.1.6. *With the notation and assumptions of Definition 4.1.4, in particular* asymmetric *inputs, we have*

$$\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N}) \textit{ iff } \mathcal{M} \prec_I \mathcal{N}.$$

Proof. If $\mathcal{M} \prec_I \mathcal{N}$, we have $\mathcal{M}(\mathbb{L}) \prec_L \mathcal{N}(\mathbb{L})$ and so $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$ by Theorem 3.0.6 (or direct verification). Conversely, let $M_0 = I, \ldots, M_r$ and $N_0 = I, \ldots, N_s$ denote the adjacency matrices for $\mathcal{M}$ and $\mathcal{N}$ respectively. Since $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$, there exists $L = [d^q_\ell] \in M(r, s+1; \mathbb{Q})$ such that

$$M_\ell = \sum_{q=0}^{s} d^q_\ell N_q, \ \ell \in \mathbf{r}.$$

Suppose that $\mathcal{F} \in \mathcal{M}(\mathbb{L})$ has model $f : V \times V^r \to V$. Define $g : V \times V^s \to V$ by

$$g(x_0; x_1, \cdots, x_s) = f(x_0; L(x_0, \cdots, x_s)).$$

In order to prove $f <^{\imath}_{(L,\mathfrak{m},\mathfrak{n})} g$, it suffices to verify that

$$L(x_j; x_{\mathfrak{n}^j_1}, \cdots, x_{\mathfrak{n}^j_s}) = (x_{\mathfrak{m}^j_1}, \cdots, x_{\mathfrak{m}^j_r}),\ j \in \mathbf{n}. \tag{4.1.1}$$

Let $j \in \mathbf{n}$, we have

$$m^{\ell}_{ij} = \sum_{q=0}^{s} d^q_{\ell} n^q_{ij},\ i \in \mathbf{n}.$$

Multiply this equation by x_i and sum over i to obtain

$$\begin{aligned} \sum_{i=1}^{n} m^{\ell}_{ij} x_i &= \sum_{i=1}^{n} \sum_{q=0}^{s} d^q_{\ell} n^q_{ij} x_i \\ &= d^0_{\ell} \sum_{i=1}^{n} n^0_{ij} x_i + \sum_{q=1}^{s} d^q_{\ell} \sum_{i=1}^{n} n^q_{ij} x_i. \end{aligned}$$

By the definition of connection matrix [2] and Equation 2.1.1, we have $\sum_{i=1}^{n} m^{\ell}_{ij} x_i = x_{\mathfrak{m}^j_{\ell}}$, $\ell \in \mathbf{r}$, $\sum_{i=1}^{n} n^0_{ij} x_i = x_j$, and $\sum_{i=1}^{n} n^q_{ij} x_i = x_{\mathfrak{n}^j_q}$, $q \in \mathbf{s}$. Hence

$$x_{\mathfrak{m}^j_{\ell}} = d^0_{\ell} x_j + \sum_{q=1}^{s} d^q_{\ell} x_{\mathfrak{n}^j_q},\ \ell \in \mathbf{r}.$$

Since the ℓth component of $L(x_j; x_{\mathfrak{n}^j_1}, \cdots, x_{\mathfrak{n}^j_s})$ is $d^0_{\ell} x_j + \sum_{q=1}^{s} d^q_{\ell} x_{\mathfrak{n}^j_q}$, we have proved (4.1.1), and so $\mathcal{M} \prec_I \mathcal{N}$. □

Remark 4.1.7. Using the same proof, Lemma 4.1.6 holds for scalar signalling networks $\mathcal{M}(\mathbb{S})$ (cf. Example 2.1.14).

Proposition 4.1.8. *(Notation of Definitions 4.1.4, 2.1.13.) $\mathcal{N} \sim_I \mathcal{M}$ iff $\mathcal{N}(\mathbb{L}) \sim \mathcal{M}(\mathbb{L})$ iff $\mathcal{N}(\mathbb{S}) \sim \mathcal{M}(\mathbb{S})$.*

[2] see Section 2.1.1

Proof. If $\mathcal{N} \sim_I \mathcal{M}$ then obviously $\mathcal{N}(\mathbb{S}) \sim \mathcal{M}(\mathbb{S})$ and $\mathcal{N}(\mathbb{L}) \sim \mathcal{M}(\mathbb{L})$. Conversely, if either $\mathcal{N}(\mathbb{S}) \sim \mathcal{M}(\mathbb{S})$ or $\mathcal{N}(\mathbb{L}) \sim \mathcal{M}(\mathbb{L})$, then we have $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ (Theorem 3.0.6) and so $\mathcal{N} \sim_I \mathcal{M}$ by Lemma 4.1.6. $\square$

Remark 4.1.9. Lemma 4.1.6 and Proposition 4.1.8 in general *fail* for networks which have (some) symmetric inputs, discussed in the next chapter. However, there is no difficulty in extending the results to networks which have more than one class of cell as long as the inputs are asymmetric. In particular, the algebraic condition $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ is required to hold for each cell class.

4.2 Output equivalence

Output equivalence is applicable to coupled cell networks with general phase space (manifold) and continuous models as well as some classes of discrete systems. The basic idea is that a network $\mathcal{N}$ 'output dominates' a network $\mathcal{M}$ if given a system $\mathcal{F} \in \mathcal{M}$, we can find a system $\mathcal{G} \in \mathcal{N}$ which has identical dynamics to $\mathcal{F}$ and is such that each cell in $\mathcal{G}$ is built from several cells in $\mathcal{F}$ together with a single Add-Subtract cell and Scaling cells acting on (un-integrated) outputs (see Figure 2.5(b)). We start with a simple example to illustrate the ideas.

Example 4.2.1. Let $\mathcal{M},\mathcal{N}$ be the network architectures of Example 4.1.1 (see Figure 4.1). We show $\mathcal{M} \prec \mathcal{N}$ by combining outputs rather than inputs. Suppose that the model for $\mathcal{F} \in \mathcal{M}$ is

$$\begin{aligned} x_1' &= f(x_1; x_1, x_2), \\ x_2' &= f(x_2; x_1, x_2). \end{aligned}$$

We look for a model g for $\mathcal{G} \in \mathcal{N}$, $\mathcal{F} \prec \mathcal{G}$, such that

$$g(x_0; x_1, x_2) = \sum_{\gamma} c_\gamma f_\gamma(x_0; x_1, x_2),$$

where the sum is over all maps $\gamma : \{1,2\} \to \{0,1,2\}$, $f_\gamma(x_0; x_1, x_2) = f(x_0; x_{\gamma(1)}, x_{\gamma(2)})$, and $c_\gamma \in \mathbb{Q}$. In order that the dynamics of $\mathcal{G}$ is identical to that of $\mathcal{F}$, it suffices that $g(x; x, y) = f(x; x, y)$ and $g(x; y, y) = f(x; y, x)$. A straightforward computation shows that there is a two parameter family of solutions for g given by

$$\begin{aligned} g(x_0; x_1, x_2) \quad = \quad & \alpha f(x_0; x_1, x_2) + (\alpha - 1) f(x_0; x_0, x_1) + \\ & (1 - \alpha) f(x_0; x_0, x_2) + (1 - \beta) f(x_0; x_1, x_0) + \\ & (\beta - \alpha) f(x_0; x_1, x_1) + \beta f(x_0; x_2, x_0) - \beta f(x_0; x_2, x_1), \end{aligned}$$

where $\alpha, \beta \in \mathbb{Q}$. If we take $\alpha = 1$, $\beta = 0$, we get

$$g(x_0; x_1, x_2) = f(x_0; x_1, x_2) + f(x_0; x_1, x_0) - f(x_0; x_1, x_1),$$

which should be compared with the solution found in Example 4.1.1. We can build the system $\mathcal{G}$ so as to realize the dynamics of $\mathcal{F}$ using Add-Subtract cells together with the cell used in $\mathcal{F}$. See Figure 4.3 (and also Figure 2.5(b)). Observe that the system shown in Figure 4.3 can be transformed back into the system $\mathcal{F}$ by removing the outer triangles defining cells of class **B** and then cancelling outputs using linearity (of vector fields). Note that unlike the input based analysis of Example 4.1.1, this configuration works whatever the phase space.

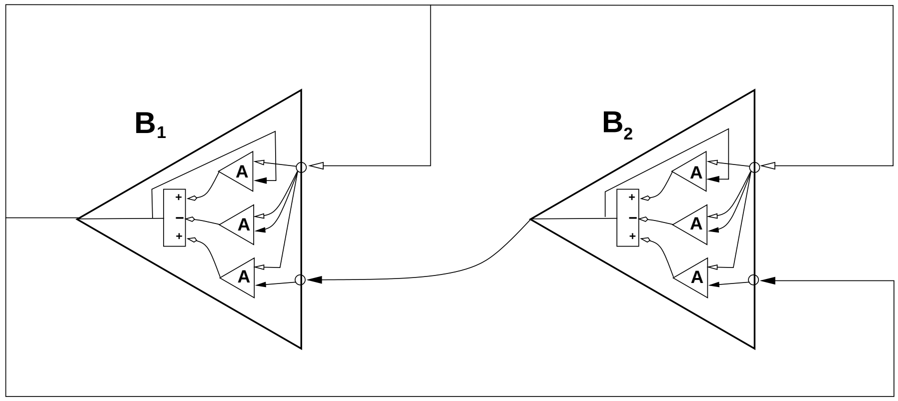


Figure 4.3: The system $\mathcal{G}$ built using 3 cells from $\mathcal{F}$ and an Add-Subtract cell.

We now formalize the concepts of output domination and equivalence. For simplicity, we work with the case of asymmetric inputs. However, the definitions extend easily and transparently to allow for symmetric inputs, discussed in the next chapter.

Suppose that M is a smooth manifold and $f : M \times M^r \to TM$, $g : M \times M^s \to TM$ are smooth families of vector fields on M, $r, s \in \mathbb{Z}^+$. Let $\mathbf{A}(r, s)$ denote the set of all maps $\gamma : \{1, \ldots, r\} \to \{0, \ldots, s\}$. If $\gamma \in \mathbf{A}(r, s)$, define $f_\gamma : M \times M^s \to TM$ by

$$f_\gamma(x_0; x_1, \ldots, x_s) = f(x_0; x_{\gamma(1)}, \ldots, x_{\gamma(r)}),\ (x_0, (x_1, \ldots, x_s)) \in M \times M^s).$$

(Addition is in $T_{x_0}M$.)

Definition 4.2.2. Let M be a vector space, $n \geq 1$, and r, s be non-negative integers. Suppose that $f : M \times M^r \to TM$, $g : M \times M^s \to TM$ are families of vector fields, $\mathfrak{m} = [\mathfrak{m}^1, \ldots, \mathfrak{m}^n] \in M(r, n; \mathbb{Z})$, $\mathfrak{n} = [\mathfrak{n}^1, \ldots, \mathfrak{n}^n] \in M(s, n; \mathbb{Z})$ are connection matrices and $C : \mathbf{A}(r, s) \to \mathbb{Q}$. We say f is $(C, \mathfrak{m}, \mathfrak{n})$-*output dominated by* g, written $f <^O_{(C,\mathfrak{m},\mathfrak{n})} g$, if

1. $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$ (as a sum of vector fields on M).

2. For $j \in \mathbf{n}$ we have $g(x_j, x_{\mathfrak{n}_1^j}, \cdots, x_{\mathfrak{n}_s^j}) = f(x_j; x_{\mathfrak{m}_1^j}, \cdots, x_{\mathfrak{m}_r^j})$.

Remarks 4.2.3. (1) Just as for input domination, a *necessary* condition for output domination is $\{\mathfrak{m}_1^j, \ldots, \mathfrak{m}_r^j\} \subset \{j, \mathfrak{n}_1^j, \ldots, \mathfrak{n}_s^j\}$, $j \in \mathbf{n}$. (2) We say f is *output dominated* by g if $f <^O_{(C,\mathfrak{m},\mathfrak{n})} g$ for some choice of C.
(3) It is possible to extend Definition 4.2.2 to apply to discrete systems defined on compact M provided that $f : M \times M^r \to M$ is sufficiently C^0-close to the projection $\pi(x_0; x_1, \ldots, x_r) = x_0$. For this, we may use the exponential map of a Riemannian metric on M so as to define uniform local linear coordinate systems at every point of M.

Definition 4.2.4. Let $\mathcal{M}$ and $\mathcal{N}$ be coupled identical cell networks such that

(a) $n(\mathcal{M}) = n(\mathcal{N}) = n$.

(b) Cells in $\mathcal{M}$ have r inputs, cells in $\mathcal{N}$ have s inputs.

(c) If we fix an ordering $\mathbf{C}_1, \ldots, \mathbf{C}_n$ of the cells in $\mathcal{N}$, then the associated connection matrix is $\mathfrak{n} = [\mathfrak{n}^1, \ldots, \mathfrak{n}^n]$.

We write $\mathcal{M} \prec_O \mathcal{N}$ if there exist $C : \mathbf{A}(r,s) \to \mathbb{Q}$ and an ordering of the cells of $\mathcal{M}$, with associated connection matrix $\mathfrak{m}$, such that for every $\mathcal{F} \in \mathcal{M}$, there exists $\mathcal{G} \in \mathcal{N}$ for which $f <^O_{(C,\mathfrak{m},\mathfrak{n})} g$. If $\mathcal{N} \prec_O \mathcal{M}$ and $\mathcal{M} \prec_O \mathcal{N}$, we say $\mathcal{N}$ and $\mathcal{M}$ are *output equivalent* and write $\mathcal{N} \sim_O \mathcal{M}$.

Remark 4.2.5. We write $\mathcal{M} \prec_{O,\mathbb{Z}} \mathcal{N}$ if $\mathcal{M} \prec_O \mathcal{N}$ and we can choose the map C of the definition to be $\mathbb{Z}$-valued. We similarly define $\mathcal{M} \sim_{O,\mathbb{Z}} \mathcal{N}$. We have $\mathcal{M} \prec_{O,\mathbb{Z}} \mathcal{N}$ in Example 4.2.1 (indeed, $\mathcal{M} \sim_{O,\mathbb{Z}} \mathcal{N}$).

Theorem 4.2.6. *(Notation and assumptions as above.)* $\mathcal{N} \sim_O \mathcal{M}$ *iff* $\mathbf{A}(\mathcal{N}) = \mathbf{A}(\mathcal{M})$ *iff* $\mathcal{N} \sim \mathcal{M}$.

Here, we prove Theorem 4.2.6 when the cells have asymmetric inputs. The proof for the case of symmetric inputs is given in Chapter 5. We break the proof of Theorem 4.2.6 into a number of lemmas of independent interest. These lemmas also give a simple algorithm for computing an explicit output equivalence. As remarked in Chapter 2, the non-trivial part of this result is the construction of the output equivalence, granted that the algebraic condition $\mathbf{A}(\mathcal{N}) = \mathbf{A}(\mathcal{M})$ is satisfied.

Let $\mathcal{M}$ be a coupled n identical cell network and suppose that cells in $\mathcal{M}$ have $r \geq 1$ asymmetric inputs. Let $\mathbb{A}(\mathcal{M}) = \{M_0, M_1, \ldots, M_r\}$ be the set of adjacency matrices. Given $u \in \mathbf{r}$, let $\mathcal{M}^{-u}$ be the n identical cell network with $r-1$ asymmetric inputs and $\mathbb{A}(\mathcal{M}^{-u}) = \mathbb{A}(\mathcal{M}) \smallsetminus \{M_u\}$. That is, $\mathcal{M}^{-u}$ is obtained from $\mathcal{M}$ by removing the uth input from each cell.

Lemma 4.2.7. *(Notation and assumptions as above.) If* $u \in \mathbf{r}$ *then* $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M}^{-u})$ *iff* $\mathcal{M} \sim_O \mathcal{M}^{-u}$.

Proof. We start by showing that $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M}^{-u})$ implies $\mathcal{M} \sim_O \mathcal{M}^{-u}$. Permuting inputs we may and shall assume $u = r$. If $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M}^{-r})$, then

$$M_r = \sum_{i=0}^{r-1} d^i M_i, \tag{4.2.2}$$

where $d^0, \ldots, d^{r-1} \in \mathbb{Q}$. It suffices to show $\mathcal{M} \prec_O \mathcal{M}^{-r}$ (the reverse order is trivial). Suppose $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^r \rightarrow TM$. Define $g : M \times M^{r-1} \rightarrow TM$

by

$$g(x_0; x_1, \cdots, x_{r-1}) = \sum_{i=0}^{r-1} d^i f(x_0; x_1, \cdots, x_{r-1}, x_i).$$

Using (4.2.2), we show easily that if $j \in \mathbf{n}$, then

$$g(x_j, x_{\mathfrak{m}^j_1}, \cdots, x_{\mathfrak{m}^j_{r-1}}) = f(x_j; x_{\mathfrak{m}^j_1}, \cdots, x_{\mathfrak{m}^j_r}), \tag{4.2.3}$$

where $[\mathfrak{m}^1, \ldots, \mathfrak{m}^n]$ is the connection matrix for $\mathcal{M}$. If $\mathcal{G} \in \mathcal{M}^{-r}$ has model g, then f is output dominated by g. Hence $\mathcal{M} \prec_O \mathcal{M}^{-r}$. It remains to show that if $\mathcal{M} \sim_O \mathcal{M}^{-u}$ then $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M}^{-u})$. This can either be seen by reversing the previous argument or by observing that if $\mathcal{M} \sim_O \mathcal{M}^{-u}$ then certainly $\mathcal{M}(\mathbb{L}) \sim_L \mathcal{M}^{-u}(\mathbb{L})$ and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M}^{-u})$ by Theorem 3.0.6. $\square$

Lemma 4.2.8. *(Notation and assumptions as above.) If the network $\mathcal{M}^\star$ is derived from $\mathcal{M}$ by removing inputs so that*

(a) $\mathbf{A}(\mathcal{M}^\star) = \mathbf{A}(\mathcal{M})$,

(b) $\mathbb{A}(\mathcal{M}^\star)$ *is a linearly independent set (and so a basis for* $\mathbf{A}(\mathcal{M})$*),*

then $\mathcal{M}^\star \sim_O \mathcal{M}$.

Proof. The result follows by repeated application of Lemma 4.2.7. $\square$

Remark 4.2.9. For networks with *asymmetric* inputs, $\mathcal{M}^\star$ is automatically *minimal* in the sense of Aguiar & Dias [6]. That is, the number of inputs of $\mathcal{M}^\star$ is minimal. However, if we allow symmetric inputs then $\mathcal{M}^\star$ may not be minimal even if $\mathcal{M}^\star$ satisfies (a) and (b) of the lemma.

Example 4.2.10. Let $\mathcal{M}$ be the network with non-identity adjacency matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. Here, $\mathbf{A}(\mathcal{M}^\star) = \mathbf{A}(\mathcal{M})$ but $\mathcal{M}^\star$ is not minimal in the sense of Aguiar & Dias [6]. The minimal network associated to $\mathcal{M}$ has non-identity adjacency matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Lemma 4.2.11. *If* $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ *then* $\mathcal{M} \sim_O \mathcal{N}$.

Proof. It follows from Lemma 4.2.8 that we can assume that $\mathbb{A}(\mathcal{M})$, $\mathbb{A}(\mathcal{N})$ both define bases of $\mathbf{A}(\mathcal{M})$. In particular, cells in both networks have the same number of inputs. Let $\mathbb{A}(\mathcal{M}) = \{M_0, \dots, M_r\}$, $\mathbb{A}(\mathcal{N}) = \{N_0, \dots, N_r\}$. Suppose that there is exactly one $j \in \mathbf{r}$ such that $N_j \neq M_j$ (of course, $M_0 = N_0 = I$). Permuting inputs, it is no loss of generality to assume $j = r$. We prove $\mathcal{M} \prec_O \mathcal{N}$. Since $M_r \in \mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$, we may write

$$M_r = \sum_{i=0}^{r} d^i N_i, \tag{4.2.4}$$

where the coefficients $d^i \in \mathbb{Q}$ and are unique. Suppose $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^r \to TM$. If we define $g : M \times M^r \to TM$ by

$$g(x_0; x_1, \cdots, x_r) = \sum_{i=0}^{r} d^i f(x_0; x_1, \cdots, x_{r-1}, x_i),$$

then g will be the model for a system $\mathcal{G} \in \mathcal{N}$ and g will output dominate f. The proof of the reverse order $\mathcal{N} \prec_O \mathcal{M}$ is exactly the same. The general case now follows by observing that we can transform $\mathcal{M}$ into $\mathcal{N}$ by modifying one input at a time. $\square$

Remarks 4.2.12. (1) Lemmas 4.2.7, 4.2.11 give an iterative algorithm for constructing an explicit output equivalence. Even if $\mathbb{A}(\mathcal{M})$, $\mathbb{A}(\mathcal{N})$ are both bases, the output equivalence need not be unique — see Example 4.2.1.
(2) Using similar methods to those given above, we can show that if $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$ then $\mathcal{M} \prec_O \mathcal{N}$. Indeed, we may give an explicit formula that realizes the output domination. Suppose that cells in $\mathcal{M}$ have r inputs, cells in $\mathcal{N}$ have s inputs. For each $u \in \mathbf{r}$, let $M_u = \sum_{i=0}^{s} d_u^i N_i$ where $[d_u^i] \in M(r+1, s; \mathbb{Q})$. If $\mathcal{F} \in \mathcal{M}$ has model f, and we define

$$g(x_0; x_1, \cdots, x_s) = \sum_{i_1=0}^{s} \cdots \sum_{i_r=0}^{s} (\prod_{u=1}^{r} d_u^{i_u}) f(x_0; x_{i_1}, \cdots, x_{i_r}),$$

then g models the required system $\mathcal{G} \in \mathcal{N}$.

Lemma 4.2.13. $\mathcal{M} \sim_O \mathcal{N} \Longrightarrow \mathcal{M} \sim \mathcal{N} \Longrightarrow \mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$.

Proof. If $\mathcal{M} \sim_O \mathcal{N}$ then obviously $\mathcal{M} \sim \mathcal{N}$. Hence, $\mathcal{M}(\mathbb{L}) \sim \mathcal{N}(\mathbb{L})$ and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ by Theorem 3.0.6. □

Proof of Theorem 4.2.6 The proof is immediate from Lemma 4.2.11 and 4.2.13. □

Remarks 4.2.14. (1) Theorem 4.2.6 extends easily to networks containing more than one class of cell. Output equivalence holds iff we can index cells so that the condition $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ holds for each cell class. (2) Theorem 4.2.6 applies to scalar signalling networks (Definition 2.1.13) – the proof is formally identical.

4.3 Examples

4.3.1 Systems with toral phase space

In this section we consider coupled systems with phase space a torus $\mathbb{T}^q$, $q \geq 1$ (more generally, everything we say works for a phase space of the form $\mathbb{R}^p \times \mathbb{T}^q$, $q \geq 1$). Suppose that $\mathcal{M}$, $\mathcal{N}$ are coupled cell networks. It follows from Theorem 4.2.6 that $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ iff $\mathcal{M} \sim_O \mathcal{N}$. So if $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$, we always have $\mathcal{M}(\mathbb{T}) \sim_O \mathcal{N}(\mathbb{T})$. As we shortly see, this is not necessarily so if we work in terms of input equivalence. That is, $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ does *not* generally imply $\mathcal{M}(\mathbb{T}) \sim_I \mathcal{N}(\mathbb{T})$. However, if $\mathcal{M} \sim_{I,\mathbb{Z}} \mathcal{N}$, then we do have $\mathcal{M}(\mathbb{T}) \sim_I \mathcal{N}(\mathbb{T})$. This is so since adding integer multiples of angles gives a well-defined angle. Thus, the networks of Example 4.1.1 are input equivalent.

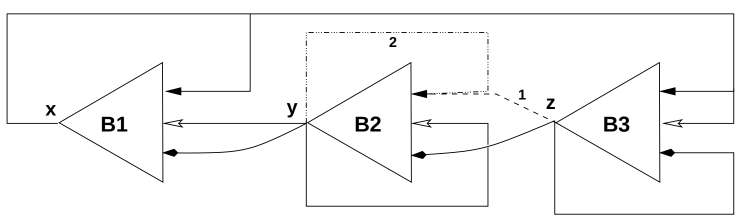


Figure 4.4: Networks $\mathcal{M}$ and $\mathcal{N}$ differ in the first input to $\mathbf{B}_2$.

Example 4.3.1. In Figure 4.4 we show two networks $\mathcal{M}$, $\mathcal{N}$ that differ only in the first input to $\mathbf{B}_2$. For the network $\mathcal{M}$, this input comes from $\mathbf{B}_3$, for $\mathcal{N}$ it comes from $\mathbf{B}_2$. The non-identity adjacency matrices for $\mathcal{M}$ are $M_0 = I$,

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The adjacency matrices for $\mathcal{N}$ are given by $N_0 = I$, $N_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $N_2 = M_2, N_3 = M_3$. It is straightforward to verify that $\mathbb{A}(\mathcal{M})$, $\mathbb{A}(\mathcal{N})$ both define a basis for $\mathbf{A}$. Moreover,

$$\begin{aligned} N_1 &= \frac{1}{2}(M_0 + M_1 + M_2 - M_3), \\ M_1 &= -N_0 + 2N_1 - N_2 + N_3, \end{aligned}$$

and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N}) = \mathbf{A}$. Suppose that $\mathcal{F} \in \mathcal{M}(\mathbb{T})$ has model f and $\mathcal{G} \in \mathcal{N}(\mathbb{T})$ has model g. We assume both systems have phase space $\mathbb{T}$ and denote the variables for $\mathcal{F}$ by θ_i and for $\mathcal{G}$ by ϕ, $i \in \mathbf{3}$. With these conventions the differential equations for $\mathcal{F}$ and $\mathcal{G}$ are given by

$$\begin{aligned} \theta_1' &= f(\theta_1; \theta_1, \theta_2, \theta_2), & \phi_1' &= g(\phi_1; \phi_1, \phi_2, \phi_2), \\ \theta_2' &= f(\theta_2; \theta_3, \theta_2, \theta_3), & \phi_2' &= g(\phi_2; \phi_2, \phi_2, \phi_3), \\ \theta_3' &= f(\theta_3; \theta_1, \theta_1, \theta_3), & \phi_3' &= g(\phi_3; \phi_1, \phi_1, \phi_3). \end{aligned}$$

Since $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$, we have $\mathcal{M} \sim \mathcal{N}$ and so $\mathcal{M} \sim_O \mathcal{N}$. In particular, if $\mathcal{F}$ and

$\mathcal{G}$ are output equivalent then an output equivalence is given by

$$\begin{aligned}
f(\theta_0;\theta_1,\theta_2,\theta_3) &= \frac{1}{2}(g(\theta_0;\theta_0,\theta_2,\theta_3)+g(\theta_0;\theta_1,\theta_2,\theta_3)\\
&\quad +g(\theta_0;\theta_2,\theta_2,\theta_3)-g(\theta_0;\theta_3,\theta_2,\theta_3)),\\
g(\phi_0;\phi_1,\phi_2,\phi_3) &= -f(\phi_0;\phi_0,\phi_2,\phi_3)+2f(\phi_0;\phi_1,\phi_2,\phi_3)\\
&\quad -f(\phi_0;\phi_2,\phi_2,\phi_3)+f(\phi_0;\phi_3,\phi_2,\phi_3).
\end{aligned}$$

We emphasize these relations are not unique. There is a system of 24 linear equations in 64 unknowns which determine the possible output equivalences, we present one solution from a 40-dimensional family. Our solution is given by the proof of Theorem 4.2.6. The question of input equivalence is more subtle. We have $\mathcal{F} \prec_I \mathcal{G}$ if and only if

$$g(\phi_0;\phi_1,\phi_2,\phi_3) = f(\phi_0;2\phi_1-\phi_0-\phi_2+\phi_3,\phi_2,\phi_3). \tag{4.3.5}$$

The input equivalence is uniquely determined and well defined since coefficients are all integers. On the other hand, $\mathcal{G} \not\prec_I \mathcal{F}$ since the relation for input equivalence has to be

$$f(\theta_0;\theta_1,\theta_2,\theta_3) = g(\theta_0;\frac{1}{2}(\theta_0+\theta_1+\theta_2-\theta_3),\theta_2,\theta_3),$$

and this is not well-defined on the torus.

If instead we consider discrete dynamical systems on $\mathbb{T}$, we see that if (4.3.5) holds then $\mathcal{M} \prec_{I,\mathbb{Z}} \mathcal{N}$ and $\mathcal{M} \prec_{O,\mathbb{Z}} \mathcal{N}$. Hence $\mathcal{M}(\mathbb{T}) \prec \mathcal{N}(\mathbb{T})$ for discrete dynamics. The converse relation is less clear. Certainly, $\mathcal{N} \not\prec_{I,\mathbb{Z}} \mathcal{M}$. It is conceivable that $\mathcal{N} \prec_{O,\mathbb{Z}} \mathcal{M}$ if there exist output equivalences in the 40-dimensional family of solutions which have integer coefficients — however, it is easy to see that there are no such solutions. Nevertheless, there remains the possibility that $\mathcal{N}(\mathbb{T}) \prec \mathcal{M}(\mathbb{T})$ for discrete dynamics. Indeed, if we assume that there exist $p \in \mathbb{Z}$ and $c \in \mathbb{T}$ such that $|g(\theta_0;\theta_1,\theta_2,\theta_3) - (c + p\theta_0)| < \pi/2$, for all $\theta_0, \ldots, \theta_3 \in \mathbb{T}$, then, using the exponential map for $\mathbb{T}^3$, we can define f as above so that g is output dominated by f. More generally, we can always continuously deform f, g to their linearizations and reduce the question to a problem of output equivalence of discrete linear systems. For example, if we take $g(\phi_0;\phi_1,\phi_2,\phi_3) = \phi_0 + \phi_1$, then it is not possible to find f realizing the same discrete dynamics as g. On the other hand if we take $g(\phi_0;\phi_1,\phi_2,\phi_3) = 2(\phi_0 + \phi_1)$, then we can find f realizing the same discrete dynamics. All of this shows that there are topological obstructions to the equivalence of $\mathcal{M}(\mathbb{T})$ and $\mathcal{N}(\mathbb{T})$ for discrete dynamics. None of these issues arise if we assume scalar signalling networks.

Theorem 4.3.2. *Let $\mathcal{M}(\mathbb{T})$ and $\mathcal{N}(\mathbb{T})$ be coupled cell networks with asymmetric inputs and discrete dynamics. Then the following are equivalent.*

1. $\mathcal{M} \prec_I \mathcal{N}$.
2. $\mathcal{M} \prec_O \mathcal{N}$.
3. $\mathbf{A}(\mathcal{M}) \subset \mathbb{Z}$*-span of* $\mathbb{A}(\mathcal{N})$.

Proof. Let $\mathbb{A}(\mathcal{M}) = \{M_0 = I, M_1\}$ and each cell in $\mathcal{N}$ has ℓ asymmetric inputs.

($3 \Rightarrow 1$): Let $M_1 = \sum_{i=0}^{\ell} \lambda_i N_i$, where $\lambda_i \in \mathbb{Z}$, $i \in \overline{\boldsymbol{\ell}}$. Define

$$g(x_0; x_1, \cdots, x_\ell) = f(x_0, \sum_{i=0}^{\ell} \lambda_i x_i).$$

Defining $g : \mathbb{T} \times \mathbb{T}^\ell \to \mathbb{T}$ as above, g is T-periodic in each variable and is well defined. Now, it is easy to check that $\mathcal{M} \prec_I \mathcal{N}$.

($2 \Rightarrow 1$): Suppose a coupled cell system with network architecture $\mathcal{M}$ is modelled by $f : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$, which is linear, that is, $f(x; y) = y$. Since $\mathcal{M} \prec_O \mathcal{N}$, there exist constants $\lambda_i \in \mathbb{R}$, $i \in \overline{\ell}$ such that

$$g(x_0; x_1, \cdots, x_\ell) = \sum_{i=0}^{\ell} \lambda_i \ f(x_0; x_i) = \sum_{i=0}^{\ell} \lambda_i \ x_i = f(x_0; \sum_{i=0}^{\ell} \lambda_i \ x_i).$$

Since g is 2π-periodic in each variable, for g to be well defined, $\lambda_i \in \mathbb{Z}$, for all $i \in \overline{\ell}$. By writing down in matrix form, it is easy to see that $M_1 = \sum_{i=0}^{\ell} \lambda_i \ N_i$.

($3 \Rightarrow 2$): As above, let $M_1 = \sum_{i=0}^{\ell} \lambda_i N_i$, where $\lambda_i \in \mathbb{Z}$, $i \in \overline{\boldsymbol{\ell}}$. Define

$$g(x_0; x_1, \cdots, x_\ell) = \sum_{i=0}^{\ell} \lambda_i f(x_0, x_i).$$

Defining $g : \mathbb{T} \times \mathbb{T}^\ell \to \mathbb{T}$ as above, g is T-periodic in each variable and is well defined. Now, it is easy to check that $\mathcal{M} \prec_O \mathcal{N}$.

($1 \Rightarrow 3$): Let $\mathcal{M} \prec_I \mathcal{N}$. Then $g(x_0; x_1, \cdots, x_\ell) = f(x_0, \sum_{i=0}^{\ell} \lambda_i x_i)$, for $\lambda_i \in \mathbb{R}$,

$i \in \overline{\boldsymbol{\ell}}$. Since g is 2π-periodic in each variable, for all $k_i \in \mathbb{Z}$,

$$\begin{aligned} g(x_0; x_1, \cdots, x_\ell) &= g(x_0; x_1 + 2k_1\pi, \cdots, x_\ell + 2k_\ell\pi) \\ &= f(x_0, \sum_{i=0}^{\ell} \lambda_i x_i + \sum_{i=0}^{\ell} 2\lambda_i k_i \pi). \end{aligned}$$

Choose $k_0 = 1$, $k_i = 0$, $i \in \boldsymbol{\ell}$, we must have $\lambda_0 \in \mathbb{Z}$, hence $\lambda_0 \in \mathbb{Z}$. Similarly, we can show that $\lambda_i \in \mathbb{Z}$, $i \in \boldsymbol{\ell}$. Thus $\mathbf{A}(\mathcal{M}) \subset \mathbb{Z}$-span of $\mathbb{A}(\mathcal{N})$.

□

Theorem 4.3.3. *Let $\mathcal{M}(\mathbb{T})$ and $\mathcal{N}(\mathbb{T})$ be coupled cell networks with asymmetric inputs and continuous dynamics. Then the following are equivalent:*

1. *$\mathcal{M} \prec_O \mathcal{N}$ if and only if $\mathbf{A}(\mathcal{M}) \subset \mathbb{A}(\mathcal{N})$.*

2. *$\mathcal{M} \prec_I \mathcal{N}$ if and only if $\mathbf{A}(\mathcal{M}) \subset \mathbb{Z}$-span of $\mathbb{A}(\mathcal{N})$.*

Proof. Let $\mathbb{A}(\mathcal{M}) = \{M_0 = I, M_1\}$ and each cell in $\mathcal{N}$ has ℓ asymmetric inputs.
(1) The statement follows from Lemma 4.2.11.
(2) ($\Leftarrow$): Let $M_1 = \sum_{i=0}^{\ell} \lambda_i N_i$, where $\lambda_i \in \mathbb{Z}$, $i \in \overline{\boldsymbol{\ell}}$. Define

$$g(x_0; x_1, \cdots, x_\ell) = f(x_0, \sum_{i=0}^{\ell} \lambda_i x_i).$$

Defining $g : \mathbb{T} \times \mathbb{T}^\ell \to \mathbb{R}$ as above, g is 2π-periodic in each variable and is well defined. Now, it is easy to check that $\mathcal{M} \prec_I \mathcal{N}$.
(2) ($\Rightarrow$): Let $\mathcal{M} \prec_I \mathcal{N}$. Then $g(x_0; x_1, \cdots, x_\ell) = f(x_0, \sum_{i=0}^{\ell} \lambda_i x_i)$, for $\lambda_i \in \mathbb{R}$,

$i \in \overline{\boldsymbol{\ell}}$. Since g is 2π-periodic in each variable, for all $k_i \in \mathbb{Z}$,

$$\begin{aligned} g(x_0; x_1, \cdots, x_\ell) &= g(x_0; x_1 + 2k_1\pi, \cdots, x_\ell + 2k_\ell\pi) \\ &= f(x_0, \sum_{i=0}^{\ell} \lambda_i x_i + \sum_{i=0}^{\ell} 2\lambda_i k_i \pi). \end{aligned}$$

Choose $k_0 = 1$, $k_i = 0$, $i \in \boldsymbol{\ell}$, we must have $\lambda_0 \in \mathbb{Z}$, hence $\lambda_0 \in \mathbb{Z}$. Similarly, we can show that $\lambda_i \in \mathbb{Z}$, $i \in \boldsymbol{\ell}$. Thus $\mathbf{A}(\mathcal{M}) \subset \mathbb{Z}$-span of $\mathbb{A}(\mathcal{N})$.

$\square$

4.3.2 Systems with non-Abelian group as a phase space

We look at two examples where the phase space is the non-Abelian Lie group SO(3) (what we say holds for any connected non-Abelian Lie group).

Example 4.3.4. We consider the networks $\mathcal{M}$ and $\mathcal{N}$ of Example 4.1.1. We choose systems $\mathcal{F} \in \mathcal{M}(\mathrm{SO}(3))$ and $\mathcal{G} \in \mathcal{N}(\mathrm{SO}(3))$. Denote the corresponding models by f and g respectively where $f, g : \mathrm{SO}(3) \times \mathrm{SO}(3)^2 \rightarrow T\mathrm{SO}(3)$. In this case we may define input equivalence using the group structure on SO(3). Specifically, if we have $g(\gamma_0; \gamma_1, \gamma_2) = f(\gamma_0; \gamma_1, \gamma_0\gamma_1^{-1}\gamma_2)$, $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{SO}(3)$, then

$$\begin{aligned} g(\gamma_1; \gamma_1, \gamma_2) &= f(\gamma_1; \gamma_1, \gamma_1\gamma_1^{-1}\gamma_2) = f(\gamma_1; \gamma_1, \gamma_2), \\ g(\gamma_2; \gamma_1, \gamma_1) &= f(\gamma_2; \gamma_1, \gamma_2\gamma_1^{-1}\gamma_1) = f(\gamma_2; \gamma_1, \gamma_2), \end{aligned}$$

and so f is input dominated by g (note that the order of the composition $\gamma_1\gamma_1^{-1}\gamma_2$

matters). We obtain the reverse relation by taking

$$g(\gamma_0; \gamma_1, \gamma_2) = f(\gamma_0; \gamma_1, \gamma_1\gamma_0^{-1}\gamma_2), \ \gamma_0, \gamma_1, \gamma_2 \in \mathrm{SO}(3).$$

Hence $\mathcal{M}(\mathrm{SO}(3)) \sim_I \mathcal{N}(\mathrm{SO}(3))$. Exactly the same arguments show that for discrete dynamics on SO(3) we have both input and output equivalence with these network structures.

Theorem 4.3.5. *Let $\mathcal{N}$ be a coupled cell network with asymmetric inputs and adjacency matrices $N_0 = I, N_1, \cdots, N_\ell$. Let $\mathcal{M}$ be a coupled cell network with asymmetric inputs and adjacency matrices $M_0 = I, M_1$. Let $n(\mathcal{M}) = \mathcal{N} = n$. Suppose there exists $N_i \neq N_j$ such that $M_1 \in \mathbb{Z}$-span of N_0, N_i, N_j. Then $\mathcal{M}(G) \prec_I \mathcal{N}(G)$, where G is a non-Abelian Lie group.*

Proof. WLOG, assume that $i = 1, j = 2$ and

$$M_1 = aN_0 + bN_1 + cN_2, \tag{4.3.6}$$

where $a, b, c \in \mathbb{Z}\backslash\{0\}$, $a+b+c = 1$. Also, at least one of a, b, c has to be negative and at least one of a, b, c has to be positive. It can be shown that $a > 0, b < 0, c < 0$ is only possible when $N_0 = N_1 = N_2$, which is not the case. Thus the only possibility is $a > 0, b > 0, c < 0$. It is easy to check that for each cell $k \in \mathbf{n}$, either $k = \mathfrak{n}_k^2$ or, $\mathfrak{n}_k^2 = \mathfrak{n}_k^1$ or, $k = \mathfrak{n}_k^1 = \mathfrak{n}_k^2$. Let $\mathcal{F} \in \mathcal{M}(G)$, $\mathcal{G} \in \mathcal{N}(G)$. Denote the corresponding continuous models by $f : G \times G \to TG$, $g : G \times G^\ell \to TG$. Let $c = -d$, where $d > 0$. Define

$$g(x; y, z, \mathbf{Z}) = f(x; x^a z^{-d} y^b). \tag{4.3.7}$$

We can check that for each $k \in \mathbf{n}$, $x_{\mathfrak{m}_k^1} = x_k^a x_{\mathfrak{n}_k^2}^{-d} x_{\mathfrak{n}_k^1}^b$. Thus, $\mathcal{M}(G) \prec_I \mathcal{N}(G)$. The same result holds for discrete dynamical systems with these network architectures.

$\square$

Example 4.3.6. We start by considering input equivalence for the networks $\mathcal{M}, \mathcal{N}$ of Example 4.3.1 when the phase space is SO(3) and we consider continuous dynamics. Suppose that $\mathcal{F} \in \mathcal{M}(\mathrm{SO}(3))$ has model f, where $f : \mathrm{SO}(3) \times \mathrm{SO}(3)^3 \to T\mathrm{SO}(3)$. We attempt to construct a model g for $\mathcal{G} \in \mathcal{N}$ which input dominates f. For example, we can try $g(\gamma_0; \gamma_1, \gamma_2, \gamma_3) = f(\gamma_0; \gamma_2^{-1}\gamma_3\gamma_0^{-1}\gamma_1^2, \gamma_2, \gamma_3)$. We find that inputs do not match for the second cell:

$$g(\phi_2; \phi_2, \phi_2, \phi_3) = f(\phi_2; \phi_2^{-1}\phi_3\phi_2, \phi_2, \phi_3) \neq f(\phi_2; \phi_3, \phi_2, \phi_3).$$

It is easy to verify that whatever the order of composition of γ_0^{-1}, γ_1^2, γ_2^{-1}, γ_3, inputs do not match for at least one cell. Consequently, $\mathcal{M}(\mathrm{SO}(3)) \not\prec_I \mathcal{N}(\mathrm{SO}(3))$.

Similar arguments show that $\mathcal{N}(\mathrm{SO}(3)) \not\prec_I \mathcal{M}(\mathrm{SO}(3))$ and that input domination either way fails for discrete dynamics. It is not clear whether or not we have $\mathcal{N}(\mathrm{SO}(3)) \prec_O \mathcal{M}(\mathrm{SO}(3))$ for discrete dynamics though the output equivalence that works for vector fields will not work for discrete dynamics. There is nothing we can say concerning the reverse relation. In particular, for discrete dynamics, we do not know whether either of the relations $\mathcal{N}(\mathrm{SO}(3)) \prec \mathcal{M}(\mathrm{SO}(3))$, $\mathcal{M}(\mathrm{SO}(3)) \prec \mathcal{N}(\mathrm{SO}(3))$ holds let alone whether or not we have equivalence.

Chapter 5

Dynamical Equivalence : General Networks

In this chapter, we will define the notion of input and output equivalence for general networks; networks with asymmetric inputs are a particular case. We did the analysis of asymmetric inputs separately because in such networks the results are more transparent and have independent proofs. In this chapter we extend the results of Chapter 4 for general networks. The results are more surprising and interesting in this case on account of the symmetry in the input structure. As a corollary of our proofs, we obtain algorithms for transforming from one network architecture to an input or output equivalent architecture so that each cell in the second architecture is expressed in terms of cells from the first architecture and conversely. We illustrate these algorithms, as well as instances of the input and output equivalence theorems and the lemmas needed for their proof, by a number of examples. Unlike in Chapter 4, we give most examples in a form that emphasizes the relations of output or input domination and we do not usually write down explicit dynamical equations.

5.1 Output equivalence

Let $\mathcal{M}$ and $\mathcal{N}$ be coupled n identical cell networks. Denote the cells of $\mathcal{N}$ by $D_1, \cdots, D_n$ (this fixes an ordering of the cells). Suppose cells in $\mathcal{N}$ have s inputs and q input types with s_i inputs of type i, for $i \in \mathbf{q}$ ($s = \sum_{i=1}^{q} s_i$). Let $\mathbb{A}(\mathcal{N}) = \{N_0 = I,\ N_i \in M_{s_i}(n; \mathbb{Z}^+),\ i \in \mathbf{q}\}$ be the set of adjacency matrices and $\mathbf{A}(\mathcal{N})$ denote the subspace of $M(n; \mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{N})$. Let $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$ be a connection matrix for $\mathcal{N}$. In this section we always assume that $\mathfrak{n}$ is the default connection matrix [1] and so the vectors $\mathfrak{n}_i^j$ are uniquely determined by the condition $\mathfrak{n}_{i\ell}^j \leq \mathfrak{n}_{i\ell'}^j$ if $\ell \leq \ell'$. We adopt similar conventions for the network $\mathcal{M}$ but now suppose there are r inputs and p input types. Given an ordering of the cells of $\mathcal{M}$, we let $\mathbb{A}(\mathcal{M}) = \{M_0 = I,\ M_i \in M_{r_i}(n; \mathbb{Z}^+),\ i \in \mathbf{p}\}$ denote the set of adjacency matrices and $\mathbf{A}(\mathcal{M})$ denote the subspace of $M(n; \mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{M})$. Denote the associated default connection matrix of $\mathcal{M}$ by $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$.

Next we formalize the concepts of output dominance and output equivalence for networks with symmetric inputs.

Let $G_{\mathcal{N}} = \prod_{i=0}^{q} S_{s_i}$, where S_{s_i} denotes the symmetric group on s_i symbols and we have taken $s_0 = 1$ (so that $S_{s_0} = S_1$ is the trivial group consisting of the identity). We define $G_{\mathcal{M}} = \prod_{i=0}^{p} S_{r_i}$, where $r_0 = 1$.

We take the natural action of $G_{\mathcal{N}}$ on $\bar{\mathbf{s}}$ (we regard $\mathbf{s}$ as identified with $\{\mathbf{s}_1, \cdots, \mathbf{s}_q\}$

[1] see Section 2.1.1

and $\bar{\mathbf{s}} = \{0\} \cup \mathbf{s}$). Let $\mathbf{A}(r,s)$ denote the set of all maps $\gamma : \mathbf{r} \to \bar{\mathbf{s}}$. We have natural left and right actions of $G_{\mathcal{N}}$ and $G_{\mathcal{M}}$ on $\mathbf{A}(r,s)$ defined by

$$\begin{aligned} \gamma &\mapsto \sigma\gamma, \qquad \gamma \in \mathbf{A}(r,s),\ \sigma \in G_{\mathcal{N}}, \\ \gamma &\mapsto \gamma\beta, \qquad \gamma \in \mathbf{A}(r,s),\ \beta \in G_{\mathcal{M}}. \end{aligned}$$

A map $C : \mathbf{A}(r,s) \to \mathbb{Q}$ will be $G_{\mathcal{N}}$-*invariant* if $C(\gamma) = C(\sigma\gamma)$ for all $\sigma \in G_{\mathcal{N}}$.

Let M be a smooth manifold. We write points $\mathbf{X} \in M \times \prod_{i=1}^{p} M^{r_i}$ in the form $\mathbf{X} = (\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_p)$, where $\mathbf{X}_i = (x_1^i, \cdots, x_{r_i}^i)$, $i \in \mathbf{p}$. We often write x_0 rather than $\mathbf{X}_0$ as the variable belongs to a single factor rather than a product of factors. We use similar notation for points in $M \times \prod_{i=1}^{q} M^{s_i}$. Given $j \in \mathbf{n}$, $i \in \mathbf{p}$, we let $\mathbf{X}_{\mathfrak{m}_i^j} \in M^{r_i}$ be the variables defined by the connection vector $\mathfrak{m}^j$. We similarly define $\mathbf{X}_{\mathfrak{n}_i^j} \in M^{s_i}$ for $i \in \mathbf{q}$. $G_{\mathcal{M}}$ acts on $\mathbf{X} \in M \times \prod_{i=1}^{p} M^{r_i}$ by

$$\begin{aligned} \beta\mathbf{X} &= (\mathbf{X}_0; \beta_1\mathbf{X}_1, \cdots, \beta_p\mathbf{X}_p), \\ &= (\mathbf{X}_0; x^1_{\beta_1(1)}, \cdots, x^1_{\beta_1(r_1)}, \cdots, x^p_{\beta_p(1)}, \cdots, x^p_{\beta_p(r_p)}), \end{aligned}$$

for $\beta = (\beta_1, \cdots, \beta_p) \in G_{\mathcal{M}} = \prod_{i=0}^{p} S_{r_i}$. $G_{\mathcal{N}}$ similarly acts on $\mathbf{X} \in M \times \prod_{i=1}^{q} M^{s_i}$.

Let $f : M \times \prod_{i=1}^{p} M^{r_i} \to TM$ be a family of $G_{\mathcal{M}}$-invariant vector fields on the smooth manifold M. For $\gamma \in \mathbf{A}(r,s)$, define $f_\gamma : M \times \prod_{i=1}^{q} M^{s_i} \to TM$ by

$$f_\gamma(x_0; x_1, \cdots, x_s) = f(x_0; x_{\gamma(1)}, \cdots, x_{\gamma(r)}),$$

where $(x_0; x_1, \cdots, x_s) \in M \times \prod_{i=1}^{q} M^{s_i}$.

Definition 5.1.1. (Notation and assumptions as above.) Suppose that $f : M \times \prod_{i=1}^{p} M^{r_i} \to TM$ is $G_{\mathcal{M}}$-invariant, $g : M \times \prod_{i=1}^{q} M^{s_i} \to TM$, and $C : \mathbf{A}(r,s) \to \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant. We say that f is $(C, \mathfrak{m}, \mathfrak{n})$-*output dominated by* g, written $f <^{O}_{(C,\mathfrak{m},\mathfrak{n})} g$, if

1. $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.

2. For $j \in \mathbf{n}$ we have $g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j})$.

Remark 5.1.2. Since C is $G_{\mathcal{N}}$-invariant, $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$ is automatically $G_{\mathcal{N}}$-invariant, even if f is not $G_{\mathcal{M}}$-invariant. We use this remark below to obtain a useful simplification of the formula $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.

The next three lemmas (Lemmas 5.1.4, 5.1.5, 5.1.6) show that the number of terms in the relation between g and f can be reduced using the $G_{\mathcal{M}}$-invariant property of f. The reduced relation obtained will define a new $G_{\mathcal{N}}$-invariant map $\hat{C}$. Before we state and prove these lemmas, it may be helpful to illustrate the ideas by means of a simple example.

Example 5.1.3. Let the single input type networks $\mathcal{M}$ and $\mathcal{N}$ have non-identity adjacency matrices $M_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $N_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ respectively. As usual, $M_0 = N_0 = I$. We have $M_1 = I + N_1$. If $\mathcal{F} \in \mathcal{M}$ has model f and we define

$$g(x_0; x_1, x_2) = f(x_0; x_0, x_1, x_2), \tag{5.1.1}$$

then g models a system $\mathcal{G} \in \mathcal{N}$ with identical dynamics to $\mathcal{F}$. In this case, $G_{\mathcal{M}} = S_3$, $G_{\mathcal{N}} = \langle \sigma \rangle = S_2$, where $\sigma(x_1, x_2) = (x_2, x_1)$. Obviously, $g(x_0; \sigma(x_1, x_2)) =$

$g(x_0; x_2, x_1) = f(x_0; x_0, x_2, x_1) = f(x_0; x_0, x_1, x_2)$ and so g is $G_{\mathcal{N}}$-invariant. Following Definition 5.1.1, we may also define g by

$$g(x_0; x_1, x_2) = af(x_0; x_0, x_1, x_2) + bf(x_0; x_0, x_2, x_1),$$

where $a + b = 1, a, b \in \mathbb{R}$. Since f is $G_{\mathcal{M}}$-invariant, the expression for g is equal to that given by (5.1.1).

Lemma 5.1.4. *(Notation and assumptions as above.) If f is $G_{\mathcal{M}}$-invariant, then $f_\gamma = f_{\gamma\beta}$ for all $\beta \in G_{\mathcal{M}}$.*

Proof. The model f is $G_{\mathcal{M}}$-invariant and so we have

$$f(x_0; x_1, \cdots, x_r) = f(x_0; x_{\beta(1)}, \cdots, x_{\beta(r)}),$$

for all $\beta \in G_{\mathcal{M}}$. Hence, if $\beta \in G_{\mathcal{M}}$, $\gamma \in \mathbf{A}(r, s)$, we have

$$\begin{aligned} f_\gamma(x_0; x_1, \cdots, x_s) &= f(x_0; x_{\gamma(1)}, \cdots, x_{\gamma(r)}), \\ &= f(x_0; x_{\gamma\beta(1)}, \cdots, x_{\gamma\beta(r)}), \\ &= f_{\gamma\beta}(x_0; x_1, \cdots, x_s). \end{aligned}$$

Therefore $f_\gamma = f_{\gamma\beta}$. $\square$

Let $\tilde{\mathbf{A}}(r, s) = \mathbf{A}(r, s)/G_{\mathcal{M}}$ denote the orbit space of $\mathbf{A}(r, s)$ under the right action by $G_{\mathcal{M}}$. Since the actions of $G_{\mathcal{N}}$ and $G_{\mathcal{M}}$ on $\mathbf{A}(r, s)$ commute, the $G_{\mathcal{N}}$-action on $\mathbf{A}(r, s)$ induces a (left) $G_{\mathcal{N}}$-action on $\tilde{\mathbf{A}}(r, s)$. Although a $G_{\mathcal{N}}$-invariant map $C : \mathbf{A}(r, s) \rightarrow \mathbb{Q}$ will not generally induce a map on $\tilde{\mathbf{A}}(r, s)$, we do have a trivial converse.

Lemma 5.1.5. *(Notation and assumptions as above.) If $\tilde{C} : \tilde{\mathbf{A}}(r,s) \rightarrow \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant, then $\tilde{C}$ lifts to a $G_{\mathcal{N}} \times G_{\mathcal{M}}$-invariant map*

$$\hat{C} : \mathbf{A}(r,s) \rightarrow \mathbb{Q}.$$

We regard the orbit space $\mathbf{A}(r,s)/G_{\mathcal{M}}$ as the set of group orbits for the $G_{\mathcal{M}}$-action on $\mathbf{A}(r,s)$. It is convenient to fix a subset $R = \{\gamma \in \mathbf{A}(r,s)\}$ such that the $\{G_{\mathcal{M}}\gamma \mid \gamma \in R\}$ partitions $\mathbf{A}(r,s)$. That is, $\cup_{\gamma \in R} G_{\mathcal{M}}\gamma = \mathbf{A}(r,s)$ and $G_{\mathcal{M}}\gamma \cap G_{\mathcal{M}}\nu \neq \emptyset$ iff $\gamma = \nu$.

Lemma 5.1.6. *(Notation as above.) Suppose that f is $G_{\mathcal{M}}$-invariant and $C : \mathbf{A}(r,s) \rightarrow \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant. Then there exists a $G_{\mathcal{N}} \times G_{\mathcal{M}}$-invariant map $\hat{C} : \mathbf{A}(r,s) \rightarrow \mathbb{Q}$ such that*

$$\sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma = \sum_{\gamma \in R} \hat{C}(\gamma) f_\gamma.$$

Proof. We have

$$\sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma = \sum_{\gamma \in R} \left(\sum_{\tau \in G_{\mathcal{M}}\gamma} C(\tau) f_\tau \right).$$

By Lemma 5.1.4, $f_\tau = f_\nu$ for all $\tau, \nu \in G_{\mathcal{M}}\gamma$. Letting $[\gamma] \in \tilde{\mathbf{A}}(r,s)$ denote the coset defined by γ, we define $\tilde{C}([\gamma]) = \sum_{\tau \in G_{\mathcal{M}}\gamma} C(\tau)$, $\gamma \in R$. This defines a $G_{\mathcal{N}}$-invariant map $\tilde{C} : \tilde{\mathbf{A}}(r,s) \rightarrow \mathbb{Q}$. Let $\hat{C} : \mathbf{A}(r,s) \rightarrow \mathbb{Q}$ be the $G_{\mathcal{N}} \times G_{\mathcal{M}}$-invariant lift given by Lemma 5.1.5. $\square$

Remark 5.1.7. In the lemmas and examples in this chapter, the lifted map $\hat{C}$ will be used to define the output relation between f and g.

Definition 5.1.8. Let $\mathcal{M}$ and $\mathcal{N}$ be coupled identical cell networks such that

(a) Both networks have n cells.

(b) Cells in $\mathcal{M}$ have p input types, cells in $\mathcal{N}$ have q input types.

(c) If we fix an ordering of the cells in $\mathcal{N}$, then the associated connection matrix is $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$.

We write $\mathcal{M} \prec_O \mathcal{N}$ and say $\mathcal{M}$ *is output dominated by* $\mathcal{N}$, if there exist an ordering of the cells of $\mathcal{M}$, with associated connection matrix $\mathfrak{m}$, and a $G_{\mathcal{N}}$-invariant map $C : \mathbf{A}(r,s)\to\mathbb{Q}$, such that for every $\mathcal{F} \in \mathcal{M}$, there exists $\mathcal{G} \in \mathcal{N}$ for which $f_{\mathcal{F}} <^O_{(C,\mathfrak{m},\mathfrak{n})} g_{\mathcal{G}}$. (Recall $\mathcal{F}$ is modelled by $f_{\mathcal{F}}$, and $\mathcal{G}$ is modelled by $g_{\mathcal{G}}$.) If $\mathcal{M} \prec_O \mathcal{N}$ and $\mathcal{N} \prec_O \mathcal{M}$, we say $\mathcal{N}$ and $\mathcal{M}$ are *output equivalent* and write $\mathcal{M} \sim_O \mathcal{N}$.

Remark 5.1.9. If all the inputs are asymmetric then any map $C : \mathbf{A}(r,s)\to\mathbb{Q}$ is a $G_{\mathcal{N}}$-invariant map. Therefore, the above definition just reduces to the definition of output equivalence for networks with asymmetric inputs (Definition 4.2.2).

Lemma 5.1.10. *The relation $\prec_O$ is transitive.*

Proof. Let $\mathcal{M}, \mathcal{N}, \mathcal{H}$ be coupled n identical cell networks with r, s, t inputs and p, q, u input types, respectively. Suppose $\mathcal{M} \prec_O \mathcal{H}$ and $\mathcal{H} \prec_O \mathcal{N}$. We show that $\mathcal{M} \prec_O \mathcal{N}$. Fix an ordering of cells in $\mathcal{N}$ with associated connection matrix $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$. Since $\mathcal{H} \prec_O \mathcal{N}$, it follows by the definition of output domination that we have an associated ordering of the cells of $\mathcal{H}$, connection matrix $\mathfrak{h} = [\mathfrak{h}^1, \cdots, \mathfrak{h}^n]$ and $G_{\mathcal{N}}$-invariant map $C_1 : \mathbf{A}(t,s)\to\mathbb{Q}$. If $\mathcal{K} \in \mathcal{H}$ is modelled by k, there exists $\mathcal{G} \in \mathcal{N}$ modelled by g such that

(1) $g = \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) k_\sigma$.

(2) For $j \in \mathbf{n}$ we have $g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = k(x_j; \mathbf{X}_{\mathfrak{h}_1^j}, \cdots, \mathbf{X}_{\mathfrak{h}_u^j})$.

Also, since $\mathcal{M} \prec_O \mathcal{H}$, we have an associated ordering of the cells of $\mathcal{M}$, connection matrix $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ and $G_{\mathcal{H}}$-invariant map $C_2 : \mathbf{A}(r,t) \to \mathbb{Q}$. If $\mathcal{F} \in \mathcal{M}$ is modelled by f, there exists $\mathcal{K} \in \mathcal{H}$ modelled by k such that

(3) $k = \sum_{\gamma \in \mathbf{A}(r,t)} C_2(\gamma) f_\gamma$.

(4) For $j \in \mathbf{n}$ we have $k(x_j; \mathbf{X}_{\mathfrak{h}_1^j}, \cdots, \mathbf{X}_{\mathfrak{h}_u^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j})$.

For $\sigma \in \mathbf{A}(t,s)$, set $x_i^\sigma = x_{\sigma(i)}$ for $i \in \mathbf{t}$. We have,

$$\begin{aligned}
g(x_0; x_1, \cdots, x_s) &= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) k_\sigma(x_0; x_1, \cdots, x_s) \\
&= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) k(x_0; x_{\sigma(1)}, \cdots, x_{\sigma(t)}) \\
&= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) k(x_0; x_1^\sigma, \cdots, x_t^\sigma) \\
&= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) \sum_{\gamma \in \mathbf{A}(r,t)} C_2(\gamma) f_\gamma(x_0; x_1^\sigma, \cdots, x_t^\sigma) \\
&= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) \sum_{\gamma \in \mathbf{A}(r,t)} C_2(\gamma) f(x_0; x_{\gamma(1)}^\sigma, \cdots, x_{\gamma(r)}^\sigma) \\
&= \sum_{\substack{\sigma \in \mathbf{A}(t,s) \\ \gamma \in \mathbf{A}(r,t)}} C_1(\sigma) C_2(\gamma) f(x_0; x_{\sigma\circ\gamma(1)}, \cdots, x_{\sigma\circ\gamma(r)}).
\end{aligned}$$

Let $\tilde{\mathbf{A}}(r,s) = \{\sigma \circ \gamma \in \mathbf{A}(r,s) \mid \gamma \in \mathbf{A}(r,t), \sigma \in \mathbf{A}(t,s)\} \subset \mathbf{A}(r,s)$. Define

$C : \mathbf{A}(r,s) \to \mathbb{Q}$ by

$$C(\phi) = \begin{cases} C_1(\sigma)C_2(\gamma) & \text{if } \phi = \sigma \circ \gamma \in \tilde{\mathbf{A}}(r,s) \\ 0 & \text{if } \phi \in \mathbf{A}(r,s) \setminus \tilde{\mathbf{A}}(r,s) \end{cases}.$$

Let $\beta \in G_{\mathcal{N}}$. We have $C(\beta(\sigma \circ \gamma)) = C((\beta\sigma) \circ \gamma) = C_1(\beta\sigma)C_2(\gamma) = C_1(\sigma)C_2(\gamma) = C(\sigma \circ \gamma)$. Therefore, C is $G_{\mathcal{N}}$-invariant and the relation between f and g given by

$$g(x_0; x_1, \cdots, x_s) = \sum_{\phi \in \tilde{\mathbf{A}}(r,s)} C(\phi) f_\phi(x_0; x_1, \cdots, x_s).$$

Hence $\mathcal{M} \prec_O \mathcal{N}$ (input matching conditions follow from (2,4)). $\square$

Example 5.1.11. Let $\mathcal{M}, \mathcal{K}, \mathcal{N}$ be single input type networks with non-identity adjacency matrices $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $K_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, $N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively. Note that $M_1 = I + K_1/2$ and $K_1 = 2N_1$. We claim that $\mathcal{M} \prec_O \mathcal{K}$ and $\mathcal{K} \prec_O \mathcal{N}$. Indeed, if $f : M \times M^2 \to TM$ is the model for the system $\mathcal{F} \in \mathcal{M}$, then $h : M \times M^2 \to TM$ defined by

$$h(x; y, z) = \frac{1}{2}(f(x; y, x) + f(x; z, x)),$$

models $\mathcal{H} \in \mathcal{K}$ with the same dynamics as $\mathcal{F} \in \mathcal{M}$. Similarly, if we define $g : M \times M \to TM$ by

$$g(x; y) = h(x; y, y),$$

then g models a system $\mathcal{G} \in \mathcal{N}$ with the same dynamics as $\mathcal{H}$. Observe that

$$g(x;y) = h(x;y,y) = \frac{1}{2}(f(x;y,x) + f(x;y,x)) = f(x;y,x).$$

It is easy to check that g models a system $\mathcal{G} \in \mathcal{N}$ with the same dynamics as $\mathcal{F} \in \mathcal{M}$ and so $\mathcal{M} \prec_O \mathcal{N}$.

Before we give the main result of this section, we state and prove a useful result about output domination (an analogous result holds for input domination – see Lemma 5.2.3). We continue with our assumptions on $\mathcal{M}$ and $\mathcal{N}$ and assume that we have fixed an ordering of the cells in $\mathcal{N}$. Given an ordering of the cells in $\mathcal{M}$, denote the associated set of adjacency matrices by $M_0, M_1, \cdots, M_p$. For $j \in \mathbf{p}$, let $\mathcal{M}_j$ denote the n-cell network with one input type and adjacency matrices $\{M_0, M_j\}$. Denote the connection matrix associated to $\{M_0, M_j\}$ by $\mathfrak{m}_j$.

Lemma 5.1.12. *(Notation and assumptions as above). The following conditions are equivalent.*

1. $\mathcal{M} \prec_O \mathcal{N}$.

2. *There exists an ordering of the cells in* $\mathcal{M}$ *such that* $\mathcal{M}_j \prec_O \mathcal{N}$*, for all* $j \in \mathbf{p}$.

Proof. Suppose first that $\mathcal{M} \prec_O \mathcal{N}$. By definition of output domination, we have an associated ordering of the cells of $\mathcal{M}$, connection matrix $\mathfrak{m}$ and $G_{\mathcal{N}}$-invariant map $C : \mathbf{A}(r,s) \rightarrow \mathbb{Q}$. If $\mathcal{F} \in \mathcal{M}$ has model f, there exists $\mathcal{G} \in \mathcal{N}$ with model g such that

(1) $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.

(2) For $j \in \mathbf{n}$ we have $g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j})$.

Now suppose that f depends only on the variables $(x_0, \mathbf{X}_j) \in M \times M^{s_j}$. Then the associated system can be identified with a system in $\mathcal{M}_j$. The input matching condition (2) implies trivially that we have the correct input matching for the connection matrix $\mathfrak{m}_j$ of $\mathcal{M}_j$. Hence $\mathcal{M}_j \prec_O \mathcal{N}$. Conversely, suppose that there exists an ordering of the cells in $\mathcal{M}$ such that $\mathcal{M}_j \prec_O \mathcal{N}$, for all $j \in \mathbf{p}$. For each $j \in \mathbf{p}$, there exists a $G_{\mathcal{N}}$-invariant map $C_j : \mathbf{A}(r_j, s) \rightarrow \mathbb{Q}$ such that if f^j is the model for $\mathcal{F}_j \in \mathcal{M}_j$, there exists $\mathcal{G}_j \in \mathcal{N}$ with model g^j such that

$$g^j = \sum_{\gamma \in \mathbf{A}(r_j, s)} C_j(\gamma) f^j_\gamma,$$

and the input matching conditions hold (with $\mathfrak{m}$ replaced by $\mathfrak{m}_j$). Now suppose $\mathcal{F} \in \mathcal{M}$ has model f. We define g by

$$g = \sum_{\gamma_1 \in \mathbf{A}(r_1, s)} \cdots \sum_{\gamma_p \in \mathbf{A}(r_p, s)} C_1(\gamma_1) \cdots C_p(\gamma_p) f_{\gamma_1 \cdots \gamma_p}, \tag{5.1.2}$$

where we define $f_{\gamma_1 \cdots \gamma_p}$ by making the natural identification between $\prod_{j=1}^p \mathbf{A}(r_j, s)$ and $\mathbf{A}(r, s)$ (that is, using the identification of $\mathbf{r}$ and $\{\mathbf{r}_1, \cdots, \mathbf{r}_p\}$). It is straightforward to verify that g does define a system $\mathcal{G} \in \mathcal{N}$ which satisfies the input matching conditions (2). $\square$

Theorem 5.1.13. *(Notation as above.)* $\mathcal{M} \sim_O \mathcal{N}$ *iff* $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ *iff* $\mathcal{M} \sim \mathcal{N}$.

In order to prove Theorem 5.1.13 it suffices to show that

(A) $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N}) \Longrightarrow \mathcal{M} \prec_O \mathcal{N}$.

(B) $\mathcal{M} \prec_O \mathcal{N} \Longrightarrow \mathcal{M} \prec \mathcal{N}$.

(C) $\mathcal{M} \prec_O \mathcal{N} \Longrightarrow \mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$.

(D) $\mathcal{M} \prec \mathcal{N} \Longrightarrow \mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$.

Statement (B) is trivial. We prove (C,D) by reducing to the case of linear vector fields. Most of the work involves the proof of (A) and we start with the proof of (A) and conclude with the proofs of (C,D).

We break the proof of (A) into a number of lemmas. These lemmas also give an algorithm for computing an explicit output equivalence or domination. Throughout we assume that $\mathcal{M}$, $\mathcal{N}$ are identical cell networks and follow our established notational conventions. In particular, we assume given orderings of the cells of $\mathcal{M}$, $\mathcal{N}$ and associated adjacency and connection matrices and the inclusion $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$. The result extends to non-identical cell networks by applying the proof cell class by cell class (see Chapter 4 and note that the linear equivalence results in [18] apply to networks with multiple cell classes).

Lemma 5.1.14. *If $p = q$, and $M_i = N_i$, $i \notin \{a, b\}$, $N_a = M_b$, $N_b = M_a$ then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. If $a = b$, there is nothing to prove. Suppose without loss of generality that $a < b$. We have $r_i = s_i, i \in \mathbf{p} \smallsetminus \{a, b\}$, $r_a = s_b, r_b = s_a$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times \prod_{i=1}^{p} M^{r_i} \to TM$. Define $g : M \times \prod_{i=1}^{p} M^{s_i} \to TM$ by

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_a, \cdots, \mathbf{X}_b, \cdots, \mathbf{X}_p) = f(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_b, \cdots, \mathbf{X}_a, \cdots, \mathbf{X}_p).$$

It is easy to check that g defines the required system $\mathcal{G} \in \mathcal{N}$. □

Remark 5.1.15. As a consequence of Lemma 5.1.14, we see that if the adjacency matrices of $\mathcal{M}$ are a permutation of those of $\mathcal{N}$, then $\mathcal{M} \sim_O \mathcal{N}$.

Lemma 5.1.16. *Let $p = 2$, and $M_1 = \sum_{i \in A} \alpha_i N_i$, $M_2 = \sum_{j \in B} \epsilon_j N_j$, where $A, B \subset \overline{\mathbf{q}}$, and $\alpha_i, \epsilon_j \in \mathbb{N}$, $i \in A, j \in B$. Then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Suppose that $A = \{a_1, \cdots, a_u\}, B = \{b_1, \cdots, b_w\} \subset \overline{\mathbf{q}}$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times \prod_{i=1}^{2} M^{r_i} \to TM$. Define $g : M \times \prod_{i=1}^{q} M^{s_i} \to TM$ by

$$g(\mathbf{X}_0; \mathbf{X}_1, .., \mathbf{X}_k) = f(\mathbf{X}_0; \overline{\mathbf{X}_{a_1}^{\alpha_1}, \cdots, \mathbf{X}_{a_u}^{\alpha_u}}, \overline{\mathbf{X}_{b_1}^{\epsilon_1}, \cdots, \mathbf{X}_{b_w}^{\epsilon_w}}),$$

where $\mathbf{X}_i \in M^{s_i}$ (variables corresponding to inputs of type i, $i \in \mathbf{q}$) and $\mathbf{X}_i^\alpha$ denotes $\mathbf{X}_i$ repeated α times. It is straightforward to check that g defines the required system $\mathcal{G} \in \mathcal{N}$. $\square$

Example 5.1.17. (Illustration of Lemma 5.1.16) Let $\mathcal{N}$ be the network with non-identity adjacency matrices $N_1 = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$, $N_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and $\mathcal{Q}$ be the network with non-identity adjacency matrices $P = \begin{pmatrix} 4 & 0 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.
It is straightforward to check $P = N_1 + N_2$, $Q = 2N_0$ and so $\mathbf{A}(\mathcal{Q}) \subseteq \mathbf{A}(\mathcal{N})$. Suppose that $\mathcal{F} \in \mathcal{Q}$ has model $h : M \times M^6 \times M^2 \to TM$. Following Lemma 5.1.16, we define $g : M \times M^4 \times M^2 \to TM$ by

$$g(x_0; \overline{x_1, \cdots, x_4}, \overline{x_5, x_6}) = h(x_0; \overline{x_1, \cdots, x_4, x_5, x_6}, \overline{x_0, x_0}).$$

It can be easily checked that g models the required system $\mathcal{G} \in \mathcal{N}$.

The next two lemmas handle the most difficult cases of output domination.

Lemma 5.1.18. *Let $p = 1$ and suppose $M_1 = N_1 - N_2$ then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Set $r_1 = r$, $s_2 = \tilde{s}$ so that $s_1 = r + \tilde{s}$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^r \to TM$. Set $\mathbf{Z} = (\mathbf{X}_3, \cdots, \mathbf{X}_q) \in \prod_{i=3}^q M^{s_i}$ (the variables represented by $\mathbf{Z}$ play no role in what follows).
Define $g : M \times \prod_{i=1}^q M^{s_i} \to TM$ by

$$\begin{aligned} &g(x_0; \overline{x_1, \cdots, x_{r+\tilde{s}}}, \overline{y_1, \cdots, y_{\tilde{s}}}, \mathbf{Z}) \\ &\qquad\qquad = \sum_{i=0}^{r} (-1)^i \sum_{\mathcal{C}_i} f(x_0; \overline{y_1^{a_1}, \cdots, y_{\tilde{s}}^{a_{\tilde{s}}}, x_{j_1}, \cdots, x_{j_{r-i}}}), \end{aligned} \tag{5.1.3}$$

where $\mathcal{C}_i$ is the set of all $(\tilde{s} + r - i)$-tuples $(a_1, \cdots, a_{\tilde{s}}, j_1, \cdots, j_{r-i})$ satisfying $a_1 + \cdots + a_{\tilde{s}} = i$, $1 \le j_1 < \cdots < j_{r-i} \le r + \tilde{s}$.

Let $x_{r+i} = y_i$, $i = 1, \cdots, \tilde{s}$. It suffices to show that

$$g(x_0; \overline{x_1, \cdots, x_{r+\tilde{s}}}, \overline{y_1, \cdots, y_{\tilde{s}}}, \mathbf{Z}) = f(x_0; x_1, \cdots, x_r).$$

Suppose $t \in \mathbf{r}$ and $b_1, \cdots, b_{\tilde{s}} \in \mathbb{Z}^+$ satisfy $\sum_{i=1}^{\tilde{s}} b_i = t$. We find the coefficient of $f(x_0; \overline{y_1^{b_1}, \cdots, y_s^{b_{\tilde{s}}}, x_{j_1}, \cdots, x_{j_{r-t}}})$, where $j_v \in \mathbf{r}$, $v \in \mathbf{r} - \mathbf{t}$.
Let $(b_1, \cdots, b_{\tilde{s}}, j_1, \cdots, j_{r-t}) \in \mathcal{C}_t$ and m denote the number of b_i that are greater than equal to 1. We find that $f(x_0; \overline{y_1^{b_1}, \cdots, y_{\tilde{s}}^{b_{\tilde{s}}}, x_{j_1}, \cdots, x_{j_{r-t}}})$ appears in the sum for g when $t - m \le i \le t$ and has coefficient $(-1)^i \binom{m}{t-i}$. Hence, the coefficient of this term is $\sum_{i=t-m}^{t} (-1)^i \binom{m}{t-i}$. This is zero unless $m = 0$ ($t = 0$), in which case the

coefficient is 1 and we get $f(x_0; x_1, \cdots, x_r)$. Hence g defines the required system $\mathcal{G} \in \mathcal{N}$. $\square$

Remark 5.1.19. 1. Another way to write Equation 5.1.3 is

$$\begin{aligned} &g(x_0; \overline{x_1, \cdots, x_{r+\tilde{s}}}, \overline{y_1, \cdots, y_{\tilde{s}}}, \mathbf{Z}) \\ &\quad = \sum_{i=0}^{r} (-1)^i \sum_{\substack{1 \le j_1 < \cdots < j_{r-i} \le r+\tilde{s} \\ 1 \le k_1 \le \cdots \le k_i \le \tilde{s}}} f(x_0; \overline{y_{k_1}, \cdots, y_{k_i}, x_{j_1}, \cdots, x_{j_{r-i}}}). \end{aligned} \tag{5.1.4}$$

2. The following combinatorial identity follows from Equation 5.1.4.

$$\sum_{i=0}^{r} (-1)^i \binom{\tilde{s}+i-1}{i} \binom{r+\tilde{s}}{r-i} = 1, \quad \forall\, r, \tilde{s} \in \mathbb{N}.$$

Example 5.1.20. (Illustration of Lemma 5.1.18) Let $\mathcal{Q}$ be the network of Example 5.1.17 and $\mathcal{R}$ be the network with non-identity adjacency matrix $R_1 = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$. It is straightforward to check $R_1 = P - Q$ and so $\mathbf{A}(\mathcal{R}) \subseteq \mathbf{A}(\mathcal{Q})$. Hence, by Lemma 5.1.18, we have $\mathcal{R} \prec_O \mathcal{Q}$. Suppose that $\mathcal{F} \in \mathcal{R}$ has model $e : M \times M^4 \to TM$. We construct $\mathcal{G} \in \mathcal{Q}$ with model h such that e is output dominated by h. Noting Remark 5.1.19, we define $h : M \times M^6 \times M^2 \to TM$ by

$$\begin{aligned} &h(x_0; \overline{x_1, \cdots, x_6}, \overline{x_7, x_8}) \\ &\quad = \sum_{i=0}^{4} (-1)^i \sum_{\substack{1 \le j_1 < \cdots < j_{4-i} \le 6 \\ 7 \le k_1 \le \cdots \le k_i \le 8}} e(x_0; \overline{x_{k_1}, \cdots, x_{k_i}, x_{j_1}, \cdots, x_{j_{4-i}}}). \end{aligned} \tag{5.1.5}$$

It can be easily checked that h models the required system $\mathcal{G} \in \mathcal{Q}$. We can define a new cell class **D**, built from the cells of the system $\mathcal{F}$, which realizes the dynamics of $\mathcal{F}$ when these cells are coupled according to the network architecture $\mathcal{Q}$. See Figure 5.1.

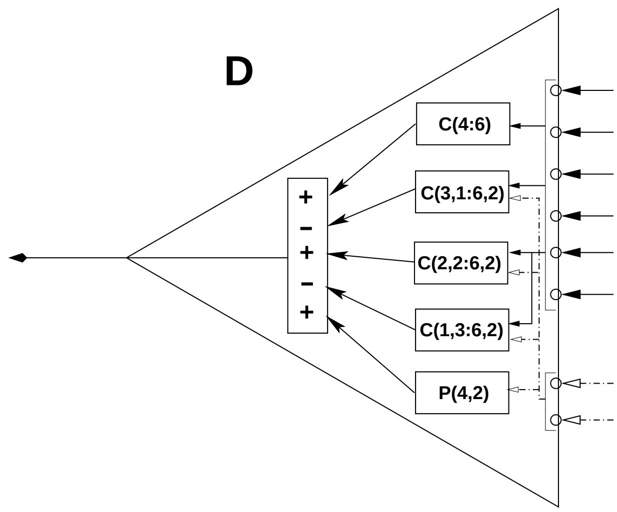


Figure 5.1: The cell **D**. The *choose and pick* cells are linear combinations of the vector field f modelling $\mathcal{F}$.

Lemma 5.1.21. *If $p = 1$ and $M_1 = \frac{1}{m}N_1$, then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Just as in the proof of Lemma 5.1.18, the variables $\mathbf{X}_j \in M^{s_j}$ play no role if $j > 1$ and so it is no loss of generality to take $p = q = 1$. The computations do not use the internal variable which we also omit. Since $p = q = 1$ and there is no internal variable, all functions will be symmetric and we omit the overline signifying symmetry. Since the case when $m = 1$ is trivial we assume $m \geq 2$. Set $r_1 = r, s_1 = s$ and note that $s = mr$. Let $\mathcal{J}$ denote the set of all tuples

$\boldsymbol{j} = (j_1, \cdots, j_u)$ of positive integers such that $j_1 \geq j_2 \geq \cdots \geq j_u \geq 1$ and $\sum_{i=1}^{u} j_i = r$. We define lexicographical ordering on $\mathcal{J}$:

$$\boldsymbol{j} = (j_1, \cdots, j_u) > \boldsymbol{j}' = (j_1', \cdots, j_{u'}'),$$

if $\exists k \in \mathbf{u}$ such that

$$j_i = j_i', \ i < k, \text{ and } j_k > j_k'.$$

Note that $\boldsymbol{j} > \boldsymbol{j}'$ does not imply $u \lesseqgtr u'$. The unique maximal and minimal elements of $\mathcal{J}$ are (r) and $(1, 1, \cdots, 1)$ respectively.

Suppose $f : M \times M^r \to TM$ models $\mathcal{F} \in \mathcal{M}$. Define $g : M \times M^{rm} \to TM$ by

$$g(x_1, \cdots, x_{rm}) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} \sum_{i_1, \cdots, i_u \in \mathbf{rm}} f(x_{i_1}^{j_1} \cdots, x_{i_u}^{j_u}), \tag{5.1.6}$$

where $c_{\boldsymbol{j}} \in \mathbb{Q}$ are constants to be determined. For fixed $\boldsymbol{j} \in \mathcal{J}$, define

$$g_{\boldsymbol{j}}(x_1, \cdots, x_{rm}) = \sum_{i_1, \cdots, i_u \in \mathbf{rm}} f(x_{i_1}^{j_1} \cdots, x_{i_u}^{j_u}).$$

Thus

$$g(x_1, \cdots, x_{rm}) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} g_{\boldsymbol{j}}(x_1, \cdots, x_{rm}).$$

We remark that each $g_{\boldsymbol{j}}$ is symmetric in $(x_1, \cdots, x_{rm})$. Hence g is symmetric in $(x_1, \cdots, x_{rm})$.

Given $\boldsymbol{j} = (j_1, \cdots, j_u) \in \mathcal{J}$, define $\mathcal{J}(\boldsymbol{j}) \subset \mathcal{J}$ to consist of all $\boldsymbol{\ell} = (\ell_1, \cdots, \ell_{u'}) \geq \boldsymbol{j}$ such that each ℓ_t can be written as a sum $\sum_{i \in I_t} j_i$, $I_t \subset \mathbf{u}$.

Suppose we are given $y_1, \cdots, y_r$ and $\boldsymbol{j} \in \mathcal{J}$. Suppose $x_1, \cdots, x_p = y_1$; $\cdots$;

$x_{(r-1)m+1}, \cdots, x_{rm} = y_r$. Then there exist strictly positive integers $A_{\boldsymbol{\ell}}^{\boldsymbol{j}}$ such that

$$g_{\boldsymbol{j}}(y_1^m, \cdots, y_r^m) = \sum_{\boldsymbol{\ell} \in \mathcal{J}(\boldsymbol{j})} A_{\boldsymbol{\ell}}^{\boldsymbol{j}} f_{\boldsymbol{\ell}}(y_1, \cdots, y_r),$$

where

$$f_{\boldsymbol{\ell}}(y_1, \cdots, y_r) = \sum f(y_{i_1}^{\ell_1}, \cdots, y_{i_{u'}}^{\ell_{u'}}),$$

and the sum is taken all distinct u'-tuples $(i_1, \cdots, i_{u'})$ of integers in $\mathbf{r}$. Each $f_{\boldsymbol{\ell}}$ is symmetric in $y_1, \cdots, y_r$. We have

$$g(y_1^m, \cdots, y_r^m) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} \left(\sum_{\boldsymbol{\ell} \in \mathcal{J}(\boldsymbol{j})} A_{\boldsymbol{\ell}}^{\boldsymbol{j}} f_{\boldsymbol{\ell}}(y_1, \cdots, y_r) \right).$$

We choose the coefficients $c_{\boldsymbol{j}}$ so that $g(y_1^m, \cdots, y_r^m) = f(y_1, \cdots, y_r)$. The term $f(y_1, \cdots, y_r)$ only occurs once in the sum we have for g (when $\boldsymbol{j}$ is the minimal element $(1, 1, 1, \cdots, 1)$ of $\mathcal{J}$). Hence $c_{(1,\cdots,1)}$ is uniquely determined. Our choice of order on $\mathcal{J}$ orders the the rows of the matrix of the linear system and our construction implies that the matrix is in upper triangular form. Hence we can solve for the coefficients $c_{\boldsymbol{j}}$. $\square$

Example 5.1.22. (Illustration of Lemma 5.1.21) Let $\mathcal{R}$ be the network of Example 5.1.20 and $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. We have $M_1 = \frac{R}{2}$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^2 \to$

TM. Following Lemma 5.1.21, define $e : M \times M^4 \to TM$ by

$$e(x_0; x_1, \cdots, x_4) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} \sum_{i_1, \cdots, i_u \in \mathbf{4}} f(x_0; x_{i_1}^{j_1}, \cdots, x_{i_u}^{j_u}),$$

where $\mathcal{J} = \{(1,1), (2)\}$ (Lemma 5.1.21) and we have omitted the overlines denoting symmetric inputs. Setting $a = c_{(1,1)}, b = c_{(2)}$, we have

$$e(x_0; x_1, \cdots, x_4) = a \sum_{i_1, i_2 \in \mathbf{4}} f(x_0; x_{i_1}, x_{i_2}) + b \sum_{i_1 \in \mathbf{4}} f(x_0; x_{i_1}, x_{i_1}). \tag{5.1.7}$$

After substituting $x_1 = x_2 = u$, $x_3 = x_4 = v$, we get the following terms: $f(x_0; u, u)$, $f(x_0; u, v)$, $f(x_0; v, v)$. The coefficient of $f(x_0; u, u)$ and $f(x_0; v, v)$ is $4a + 2b$ and the coefficient of $f(x_0; u, v)$ is $8a$. Since we require $e(x_0; u, u, v, v) = f(x_0; u, v)$, we obtain $a = \frac{1}{8}, b = -\frac{1}{4}$. It is straightforward to check that e models the required system $\mathcal{G} \in \mathcal{R}$. We can define a new cell class $\mathbf{D}$, built from the class $\mathbf{C}$ cells of the system $\mathcal{F}$, which realizes the dynamics of $\mathcal{F}$ when these cells are coupled according to the network architecture $\mathcal{R}$. See Figure 5.2.

Lemma 5.1.23. *If $p = 1$, then $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ implies $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Since $M_1 \in \mathbf{A}(\mathcal{N})$, we may write $M_1 = \sum_{i \in A} \lambda_i N_i - \sum_{i \in B} \lambda_i N_i$, where A, B are disjoint subsets of $\overline{\mathbf{q}}$ and for $i \in A \cup B$, $\lambda_i = \frac{a_i}{b_i}$, where $a_i, b_i \in \mathbb{N}$, and $(a_i, b_i) = 1$.

Let $\lambda = \mathrm{lcm}\{b_i \mid i \in A \cup B\}$ and define $\alpha_i = \lambda \lambda_i \in \mathbb{Z}^+, i \in A \cup B$. If we define $P = \sum_{i \in A} \alpha_i N_i$, $Q = \sum_{i \in B} \alpha_i N_i$, then

$$M_1 = \frac{1}{\lambda}(P - Q).$$

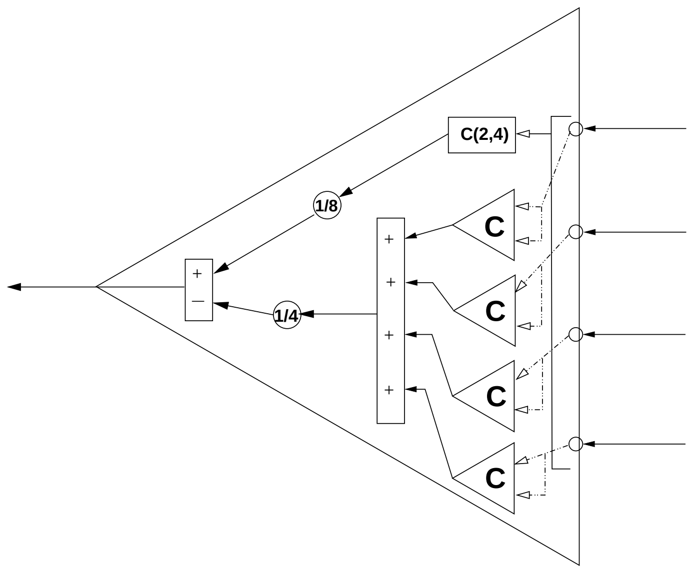


Figure 5.2: The cell **D**, built from class **C** cells

Let $\mathcal{N}_1$ be the network with adjacency matrices $\{I, P, Q\}$, and $\mathcal{M}_1$ be the network with adjacency matrices $\{I, R = P - Q\}$. Note that

1. If $Q = 0$, $\mathcal{M}_1 = \mathcal{N}_1$.

2. If $\lambda = 1$, $\mathcal{M}_1 = \mathcal{M}$.

3. If $Q = 0$ and $\lambda = 1$, $\mathcal{N}_1 = \mathcal{M}_1 = \mathcal{M}$.

We claim that

$$\mathcal{M} \prec_O \mathcal{M}_1 \prec_O \mathcal{N}_1 \prec_O \mathcal{N}.$$

Assuming the claim, the transitivity of $\prec_O$ (Lemma 5.1.10) gives $\mathcal{M} \prec_O \mathcal{N}$. The claim follows since Lemma 5.1.16 implies $\mathcal{N}_1 \prec_O \mathcal{N}$, Lemma 5.1.18 implies $\mathcal{M}_1 \prec_O \mathcal{N}_1$ and Lemma 5.1.21 implies $\mathcal{M} \prec_O \mathcal{M}_1$. $\square$

Example 5.1.24. (Illustration of Lemma 5.1.23.) Let $\mathcal{N}$ be the network defined in Example 5.1.17 and $\mathcal{M}$ be the network of Example 5.1.22. We have $M_1 = \frac{N_1}{2} + \frac{N_2}{2} - N_0$. Following the notation of the proof of Lemma 5.1.23, we have $\lambda = 2$, $P = N_1 + N_2$, $Q = 2N_0$. Note that P, Q are the non-identity adjacency matrices of the second network $\mathcal{Q}$ of Example 5.1.17. We have $\mathcal{Q} \prec_O \mathcal{N}$ (Example 5.1.17); $\mathcal{R} \prec_O \mathcal{Q}$ (Example 5.1.20), and $\mathcal{M} \prec_O \mathcal{R}$ (Example 5.1.22). Since $\prec_O$ is transitive, $\mathcal{M} \prec_O \mathcal{N}$. By using the output relations between g and h from Example 5.1.17, h and e from Example 5.1.20, and e and f from Example 5.1.22, it can be shown (see Appendix – Chapter 8.1) that the output relation between g and f is given by

$$\begin{aligned} g(x_0; \overline{x_1, \cdots, x_4}, \overline{x_5, x_6}) &= \frac{1}{4} \sum_{1 \leq j_1 < j_2 \leq 6} f(x_0; \overline{x_{j_1}, x_{j_2}}) + f(x_0; \overline{x_0, x_0}) \\ &\quad - \frac{1}{8} \sum_{1 \leq j_1 \leq 6} f(x_0; \overline{x_{j_1}, x_{j_1}}) - \frac{1}{2} \sum_{1 \leq j_1 \leq 6} f(x_0; \overline{x_0, x_{j_1}}). \end{aligned}$$

Lemma 5.1.25. *If $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$, then $\mathcal{M} \prec_O \mathcal{N}$ (statement (A) is true).*

Proof. By Lemma 5.1.12, it suffices to show that $\mathcal{M}_j \prec_O \mathcal{N}$ for all $j \in \mathbf{p}$. By Lemma 5.1.14, we may assume $j = 1$. The result follows from Lemma 5.1.23. □

Lemma 5.1.26. *If $\mathcal{M} \prec_O \mathcal{N}$ then $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ (statement (C) is true).*

Proof. Suppose $\mathcal{M} \prec_O \mathcal{N}$. The method we use is based on the linear equivalence ideas described in [18]. Specifically, we prove that $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ by restricting to the case where phase spaces equal $\mathbb{R}$ and vector fields are linear. (Notice that output domination preserves linearity of vector fields.)

Let $\mathcal{F} \in \mathcal{M}$ have (linear) model $f : \mathbb{R} \times \prod_{i=1}^{p} \mathbb{R}^{r_i} \to \mathbb{R}$. Then there exists a system $\mathcal{G} \in \mathcal{N}$ with linear model $g : \mathbb{R} \times \prod_{i=1}^{q} \mathbb{R}^{s_i} \to \mathbb{R}$ such that for each $j \in \mathbf{n}$ we have

$$g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j}), \tag{5.1.8}$$

where $\mathbf{X}_{\mathfrak{n}_i^j} = (x_{\mathfrak{n}_{i1}^j}, \cdots, x_{\mathfrak{n}_{is_i}^j})$, $i \in \mathbf{q}$, and $\mathbf{X}_{\mathfrak{n}_i^j} = (x_{\mathfrak{m}_{i1}^j}, \cdots, x_{\mathfrak{m}_{ir_i}^j})$, $i \in \mathbf{p}$. Let $k \in \mathbf{p}$ and take

$$f(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_p) = \sum_{i=1}^{r_k} x_{ki},$$

where $\mathbf{X}_v = (x_{v1}, \cdots, x_{vr_v}),\ v \in \mathbf{p}$. The corresponding g given by output domination is linear and so, noting the symmetry of inputs, we may write

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = c_{k0} x_0 + \sum_{i=1}^{q} c_{ki} \sum_{\ell=1}^{s_i} x_{i\ell},$$

where $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i}),\ i \in \mathbf{q}$, and the $c_{\alpha\beta}$ are constants. From (5.1.8) we get

$$c_{k0} x_j + \sum_{i=1}^{q} c_{ki} \sum_{\ell=1}^{s_i} x_{\mathfrak{n}_{i\ell}^j} = \sum_{i=1}^{r_k} x_{\mathfrak{m}_{ki}^j},\ j \in \mathbf{n}.$$

Putting these equations in matrix form, we obtain

$$\sum_{i=0}^{q} c_{ki} N_i = M_k.$$

Hence for each $k \in \mathbf{q}$, we have shown that $M_k \in \mathbf{A}(\mathcal{N})$ and so $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$. $\square$

Lemma 5.1.27. *If $\mathcal{M} \prec \mathcal{N}$ then $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ (statement (D) is true).*

Proof. (Sketch) Working within the class of C^1-vector fields with phase space $\mathbb{R}$, it follows that if $\mathcal{F}$ has linear model f, then there exists $\mathcal{G} \in \mathcal{N}$ with C^1-model g such that $\mathcal{G}$ has identical dynamics to $\mathcal{F}$. The statement remains true if we replace g by the derivative of g at $\mathbf{0} \in \mathbb{R} \times \mathbb{R}^q$ and then the method of proof of Lemma 5.1.26 applies (essentially we reduce to linear equivalence, cf [18]). With a little more work, we can remove the assumption that g is C^1 — identical dynamics to a linear system implies the flow is linear and from this one can show that we can always choose g to be linear. $\square$

Proof of Theorem 5.1.13. Lemmas 5.1.25, 5.1.26, 5.1.27 give statements A,C,D and, as noted previously, statement B is trivial. Interchange $\mathcal{M}$ and $\mathcal{N}$ to obtain the reverse relations. $\square$

5.2 Input equivalence

We start by giving the definition of input equivalence applicable to networks with symmetric inputs. This a straightforward extension of the definition given in the previous chapter for networks with asymmetric inputs. Aside from assuming that models are defined on vector spaces V rather than manifolds M, we closely follow the notational conventions established in the previous chapter. In particular, $\mathcal{M}$ and $\mathcal{N}$ will be coupled n identical cell networks. We fix an ordering of the cells of $\mathcal{N}$. Suppose cells in $\mathcal{N}$ have s inputs and q input types. Let $\mathbb{A}(\mathcal{N}) = \{N_0 = I,\ N_i \in M_{s_i}(n;\mathbb{Z}^+),\ i \in \mathbf{q}\}$ be the set of adjacency matrices and $\mathbf{A}(\mathcal{N})$ denote the subspace of $M(n;\mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{N})$. Let $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$ be the default connection matrix for $\mathcal{N}$.

We suppose cells in $\mathcal{M}$ have r inputs and p input types. Given an ordering of the cells of $\mathcal{M}$, we let $\mathbb{A}(\mathcal{M}) = \{M_0 = I,\ M_i \in M_{r_i}(n;\mathbb{Z}^+),\ i \in \mathbf{p}\}$ denote the set of adjacency matrices and $\mathbf{A}(\mathcal{M})$ denote the subspace of $M(n;\mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{M})$. Let $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ be the default connection matrix for $\mathcal{M}$.

Let $L = (L_1, \cdots, L_p) \in \prod_{i=1}^p M(r_i, 1 + \sum_{j=1}^q s_j; \mathbb{Q})$ and define the linear map $\mathbf{L} : V \times \prod_{i=1}^q V^{s_i} \to \prod_{i=1}^p V^{r_i}$ in the obvious (V-independent) way. Recall that $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant if there exists a homomorphism $h : G_{\mathcal{N}} \to G_{\mathcal{M}}$ such that

$$\mathbf{L}(\gamma(\mathbf{X})) = h(\gamma)\mathbf{L}(\mathbf{X}),\ \text{for all } \gamma \in G_{\mathcal{N}},\ \mathbf{X} \in V \times \prod_{i=1}^q V^{s_i}.$$

If $f : V \times \prod_{i=1}^p V^{r_i} \to V$ is $G_{\mathcal{M}}$-invariant, define $g : V \times \prod_{i=1}^q V^{s_i} \to V$ by

$$g(\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = f(\mathbf{X}_0; \mathbf{L}(\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_q)). \tag{5.2.9}$$

Since $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant, g is $G_{\mathcal{N}}$-invariant. We write $f <^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$, if

1. (5.2.9) is satisfied.

2. For $j \in \mathbf{n}$, we have $g(x_j; \mathbf{X}_{\mathfrak{n}^j_1}, \cdots, \mathbf{X}_{\mathfrak{n}^j_q}) = f(x_j; \mathbf{X}_{\mathfrak{m}^j_1}, \cdots, \mathbf{X}_{\mathfrak{m}^j_p})$.

Definition 5.2.1. (Notation and assumptions as above.) The coupled cell network $\mathcal{M}$ is *input dominated* by $\mathcal{N}$, denoted $\mathcal{M} \prec_I \mathcal{N}$, if there exist a linear map $\mathbf{L}$, an ordering of the cells of $\mathcal{M}$, with associated connection matrix $\mathfrak{m}$, such that for every $\mathcal{F} \in \mathcal{M}(\mathbb{L})$, there exists $\mathcal{G} \in \mathcal{N}(\mathbb{L})$ for which $f_{\mathcal{F}} <^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g_{\mathcal{G}}$. If $\mathcal{N} \prec_I \mathcal{M}$ and $\mathcal{M} \prec_I \mathcal{N}$, we say $\mathcal{M}$ and $\mathcal{N}$ are *input equivalent* and write $\mathcal{M} \sim_I \mathcal{N}$.

Remarks 5.2.2. (1) As we shall see later (Remark 5.2.13), the map $\mathbf{L}$ may not preserve default connection matrices. However, since inputs are symmetric, it is no loss of generality to require the default connection matrix of $\mathcal{M}$ in Definition 5.2.1. When we come to prove our main theorem, we allow for general connection matrices.

(2) We write $\mathcal{M} \prec_{I,\mathbb{Z}} \mathcal{N}$ if $\mathcal{M} \prec_I \mathcal{N}$ and we can require the entries of $\mathbf{L}$ to lie in $\mathbb{Z}$). We similarly define $\mathcal{M} \sim_{I,\mathbb{Z}} \mathcal{N}$.

Lemma 5.2.3. *(Notation and assumptions as above). The following conditions are equivalent.*

1. $\mathcal{M} \prec_I \mathcal{N}$.

2. *There exists an ordering of the cells in* $\mathcal{M}$ *such that* $\mathcal{M}_j \prec_I \mathcal{N}$, *for all* $j \in \mathbf{p}$.

Proof. The proof follows by observing that

$$f_{\mathcal{F}} <^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g_{\mathcal{G}} \iff f_{\mathcal{F}_j} <^{\imath}_{(\mathbf{L_j},\mathfrak{m}^{\mathfrak{j}},\mathfrak{n})} g_{\mathcal{G}},$$

for all $j \in \mathbf{p}$, where $\mathbf{L} = [\mathbf{L}_1, \cdots, \mathbf{L}_p]$, $\mathbf{L}_j : V \times \prod_{i=1}^{q} V^{s_i} \to V^{r_j}$, $\mathcal{F}_j \in \mathcal{M}_j$, and $\mathfrak{m}^{\mathfrak{j}}$ is the connection matrix induced on $\mathcal{M}_j$ by $\mathfrak{m}$. $\square$

As a consequence of Lemma 5.2.3, it will be no loss of generality in what follows to assume that $\mathcal{M}$ has just one input type; that is, $p = 1$. We simplify notation by setting $r_1 = r$. With these conventions, we have $G_{\mathcal{M}} = S_r \approx S_1 \times S_r$.

Suppose that the linear map $\mathbf{L} : V \times \prod_{i=1}^{q} V^{s_i} \to V^r$ is defined by the matrix $L \in M\left(r, 1 + \sum_{i=1}^{q} s_i, \mathbb{Q}\right)$. The map $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant if there exists a homomorphism $h : G_{\mathcal{N}} \to G_{\mathcal{M}} = S_r$ such that

$$\mathbf{L}(\gamma(\mathbf{X})) = h(\gamma)\mathbf{L}(\mathbf{X}),$$

for all $\gamma \in G_{\mathcal{N}}$, $\mathbf{X} \in V \times \prod_{i=1}^{q} V^{s_i}$.

Given a $G_{\mathcal{M}}$-invariant map $f : V \times V^r \to V$ and $G_{\mathcal{M},\mathcal{N}}$-equivariant linear map $\mathbf{L}$ as above, define the $G_{\mathcal{N}}$-invariant map $g : V \times \prod_{i=1}^{q} V^{s_i} \to V$ by

$$g(\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = f(\mathbf{X}_0; \mathbf{L}(\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_q)).$$

Let $L = [L_1, \cdots, L_r]$, where $L_i \in \mathbb{Q} \times \prod_{i=1}^{q} \mathbb{Q}^{s_i}$ denotes the i^{th} row of L, $i \in \mathbf{r}$. Since $\mathbf{L}(\gamma(\mathbf{X})) = h(\gamma)\mathbf{L}(\mathbf{X})$ for all $\gamma \in G_{\mathcal{N}}$, $\mathbf{X} \in V \times \prod_{i=1}^{q} V^{s_i}$, we have $[L_1, \cdots, L_r](\gamma\mathbf{X}) = h(\gamma)[L_1, \cdots, L_r](\mathbf{X})$. That is,

$$[\gamma L_1, \cdots, \gamma L_r](\mathbf{X}) = h(\gamma)[L_1, \cdots, L_r](\mathbf{X}),$$

where γL_i is defined using the natural permutation action of $G_{\mathcal{N}}$ on $\mathbb{Q} \times \prod_{j=1}^{q} \mathbb{Q}^{s_j}$, $i \in \mathbf{r}$. This is true for all $\mathbf{X}$, hence

$$[\gamma L_1, \cdots, \gamma L_r] = h(\gamma)[L_1, \cdots, L_r],$$

for all $\gamma \in G_{\mathcal{N}}$.

Definition 5.2.4. Suppose a finite group G acts on a non-empty set X. For $x \in X$, let $Gx = \{gx \mid g \in G\}$ denote the G-*orbit of* x.

Remark 5.2.5. We have $|Gx| = |G|/|G_x|$ where $G_x = \{g \in G \mid gx = x\}$ denotes the *isotropy subgroup* of G at x.

Theorem 5.2.6. *There exists* $u(\leq r) \in \mathbb{N}$, $t_1, \cdots, t_u \in \mathbb{N}$, *with* $\sum_{i=1}^{u} t_i = r$ *such that* $\{L_1, \cdots, L_r\} = \bigcup_{i=1}^{u} G_{\mathcal{N}} L^i$ *where* $|G_{\mathcal{N}} L^i| = t_i$. *(We allow* $L^i = L^j$ *for* $i \neq j$,

$i, j \in \mathbf{u}$.)

Proof. $[\gamma L_1, \cdots, \gamma L_r] = h(\gamma)\,[L_1, \cdots, L_r]$ for all $\gamma \in G_{\mathcal{N}}$. Therefore, for each $i \in \mathbf{r}$, $\gamma L_i \in \{L_1, \cdots, L_r\}$ for all $\gamma \in G_{\mathcal{N}}$. Hence, the $G_{\mathcal{N}}$-orbit of L_i is contained in $\{L_1, \cdots, L_r\}$. Suppose $L_k = L_j$ for some $k \neq j$, then $\gamma L_k = \gamma L_j$ for all $\gamma \in G_{\mathcal{N}}$. So if an element is repeated m times then its full orbit is repeated m times. Therefore, there exist $u(\leq r) \in \mathbb{N}$, L_{j_i}, for $j_1, \cdots, j_u \in \mathbf{r}$ with $|G_{\mathcal{N}} L_{j_i}| = t_i$ and $\sum_{i=1}^{u} t_i = r$ such that $\{L_1, \cdots, L_r\} = \bigcup_{j=1}^{u} G_{\mathcal{N}} L_{j_i}$. Define $L^i = L_{j_i}$, $i \in \mathbf{u}$. $\square$

Remark 5.2.7. For each $i \in \mathbf{u}$, there are t_i choices for L_{j_i}.

From now on, we write the matrix of $\mathbf{L}$ in the form

$$\left[G_{\mathcal{N}} L^1, \cdots, G_{\mathcal{N}} L^u\right].$$

That is, we group rows according to the group orbits of $G_{\mathcal{N}}$. Note that this ordering is imposing a condition on the order of inputs of $\mathcal{M}$.

Example 5.2.8. Suppose $p, q = 1, s = s_1 = 2, r = 3$, $L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Then we can take $L^1 = (0, 1, 2)$, $L^2 = (0, 1, 1)$. We have $t_1 = 2$ and $t_2 = 1$. If we write L in the form $[G_{\mathcal{N}} L^1, G_{\mathcal{N}} L^2]$, then $L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

5.2.1 Splittings and connection matrices

We recall from Chapter 4 that dynamically equivalent networks with asymmetric inputs are always input equivalent. This is not always the case for networks with symmetric inputs as we show in the next example.

Example 5.2.9. Let $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 2 & 4 \\ 2 & 0 \end{pmatrix}$ and $\mathcal{N}$ be the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 3 & 6 \\ 3 & 0 \end{pmatrix}$. We have $M_1 = \frac{2}{3}N_1$ and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$. Suppose

$$g(x_0; x_1, \cdots, x_6) = f(x_0; \mathbf{L}(x_0; x_1, \cdots, x_6)),$$

where $\mathbf{L} : V \times V^6 \to V^4$ is a $\mathcal{G}_{\mathcal{M},\mathcal{N}}$-equivariant linear map. The only possible form of L is $\begin{pmatrix} a_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 & b_3 & b_3 & b_3 & b_3 \\ a_4 & b_4 & b_4 & b_4 & b_4 & b_4 & b_4 \end{pmatrix}$. It is easy to check that there does not exist any $a_i, b_i \in \mathbb{Q}$ for which f is input dominated by g. This shows $\mathcal{M} \not\prec_I \mathcal{N}$. Similarly we can show that $\mathcal{N} \not\prec_I \mathcal{M}$ (in this case, L has two possible forms). This provides an example of network architectures $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \sim_O \mathcal{N}$ ($\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$) but $\mathcal{M} \not\prec_I \mathcal{N}$ and $\mathcal{N} \not\prec_I \mathcal{M}$.

The previous example shows that $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ is not sufficient for $\mathcal{M} \sim_I \mathcal{N}$. Note that $\mathcal{M} \sim_I \mathcal{N} \Rightarrow \mathcal{M} \sim \mathcal{N} \Rightarrow \mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$. Thus $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ is a necessary condition for $\mathcal{M} \sim_I \mathcal{N}$. In Theorem 5.2.11, we give sufficient conditions for input equivalence to hold. The sufficiency conditions come from the structure of the $\mathcal{G}_{\mathcal{M},\mathcal{N}}$-equivariant linear map $\mathbf{L}$. If $\mathcal{M} \prec_I \mathcal{N}$ and we fix a connection matrix for $\mathcal{N}$, then $\mathbf{L}$ determines a connection matrix for $\mathcal{M}$ which may not be the default connection matrix (whatever the choice of $\mathbf{L}$). In order to analyze the relationship between connection matrices of $\mathcal{M}$ and $\mathcal{N}$, we introduce the idea of splitting a valency k adjacency matrix into a sum of k valency one matrices. We find that there is a one-one correspondence between splittings and connection matrices.

Definition 5.2.10. Let $P \in M_k(n;\mathbb{Z}^+)$. A *splitting* $(P_1, \cdots, P_k)$ of P is an ordered decomposition of P into a sum $P = P_1 + \cdots + P_k$, where each $P_j \in M_1(n;\mathbb{Z}^+)$.

Suppose that the network $\mathcal{M}$ has one input type and connection matrix $\mathfrak{m}$, where $\mathfrak{m}$ is not necessarily the default connection matrix. Denote the adjacency matrices of $\mathcal{M}$ by $M_0 = I$ and M_1. The connection matrix $\mathfrak{m}$ naturally determines a unique splitting $M^1 + \cdots + M^r$ of M_1. Indeed, if we let $M^k = [m^k_{ij}]$, $k \in \mathbf{r}$, then we define $m^k_{ij} = 1$ if input k of cell j comes from cell i, else $m^k_{ij} = 0$. That is, $m^k_{ij} = 1$ iff $\mathfrak{m}^j_{1k} = i$. Conversely, every splitting of M_1 uniquely determines a connection matrix $\mathfrak{m}$ for $\mathcal{M}$. All of this applies equally well if $\mathcal{M}$ has multiple input types.

Let $\mathfrak{n}$ be a connection matrix for $\mathcal{N}$ (not necessarily the default). For $k \in \mathbf{q}$, let $\mathbf{N}_k = (N_{k1}, \cdots, N_{ks_k})$ denote the splitting of N_k naturally determined by $\mathfrak{n}$. Set $\mathbf{N} = \{\mathbf{N}_1, \cdots, \mathbf{N}_q\}$. We refer to $\mathbf{N}$ as the splitting determined by $\mathfrak{n}$.

Let $\mathbf{a} = (a_0; a_1, \cdots, a_q) \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$. We write $\mathbf{a} = (a_{ji})_{j\in\overline{\mathbf{q}}, i\in\mathbf{s_j}}$, where $a_j = (a_{j1}, \cdots, a_{js_j}) \in \mathbb{Q}^{s_j}$, $j \in \overline{\mathbf{q}}$. If $\mathbf{N} = \{\mathbf{N}_1, \cdots, \mathbf{N}_q\}$ is the set of splittings of the adjacency matrices $\{N_1, \cdots, N_q\}$ determined by $\mathfrak{n}$, then we define

$$\mathbf{a} \star \mathbf{N} = a_0 N_0 + \sum_{j=1}^{q} \sum_{i=1}^{s_j} a_{ji} N_{ji} \in M(n, n; \mathbb{Q}).$$

Theorem 5.2.11. *(Notation and assumptions as above; in particular $p = 1$.) The following statements are equivalent*

1. $\mathcal{M} \prec_I \mathcal{N}$.

2. *Suppose that $\mathfrak{n}$ is a connection matrix for $\mathcal{N}$ with associated splitting $\mathbf{N}$. There exist $u \in \mathbb{N}$, $L^i \in \mathbb{Q} \times \prod_{v=1}^{q} \mathbb{Q}^{s_v}$, $i \in \mathbf{u}$, such that $\{\mathbf{b} \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\}$ is a splitting of M_1.*

3. *There exist $u \in \mathbb{N}$, $L^i \in \mathbb{Q} \times \prod_{v=1}^{q} \mathbb{Q}^{s_v}$, $i \in \mathbf{u}$ such that for every connection matrix $\mathfrak{n}$ of $\mathcal{N}$ with associated splitting $\mathbf{N}$, $\{\mathbf{b} \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\}$ is a splitting of M_1.*

Before giving the proof of Theorem 5.2.11, we make two remarks, the first of which shows how Theorem 5.2.11 simplifies in the case of asymmetric inputs.

Remarks 5.2.12. (1) If all the inputs of the networks $\mathcal{M}$ and $\mathcal{N}$ are asymmetric then $q = s$, $s_i = 1$, $\mathbf{N} = \{N_1 = N_{11}, \cdots, \mathcal{N}_q = N_{q1}\}$ and M_1 is a splitting of itself. Thus (3) of Theorem 5.2.11 implies that there exist $u \in \mathbb{N}$, $L^i = (a_{i0}; a_{i1}, \cdots, a_{iq}) \in \mathbb{Q} \times \mathbb{Q}^q$, $i \in \mathbf{u}$ such that for every connection matrix $\mathfrak{n}$ of $\mathcal{N}$, $\{\sum_{j=0}^{q} a_{ij} N_j, i \in \mathbf{u}\}$ is a splitting of M_1. Since $M_1 \in M_1(n, \mathbb{Z})$, we must have $u = 1$. Therefore, the condition simplifies to $M_1 = \sum_{j=0}^{q} a_{ij} N_j$. Hence $M_1 \in \mathbf{A}(\mathcal{N})$; the condition obtained for networks with asymmetric inputs in Chapter 4.

(2) Condition (3) of the theorem shows that for computations, we can always take $\mathfrak{n}$ to be the default connection matrix.

Proof of Theorem 5.2.11 (1) $\Rightarrow$ (2). Suppose $\mathcal{M} \prec_I \mathcal{N}$. Then there is a linear transformation $\mathbf{L}$ with matrix $L = [G_{\mathcal{N}} L^1, \cdots, G_{\mathcal{N}} L^u]$. Let $\mathfrak{n}$ be a connection matrix for $\mathcal{N}$ and denote the corresponding splittings of $N_1, \cdots, N_q$ by $\mathbf{N}$. For each $j \in \mathbf{n}$, we have

$$\mathbf{L}(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = \mathbf{X}_{\mathfrak{m}_1^j},$$

where $\mathfrak{m}$ is a connection matrix for the network $\mathcal{M}$. Thus $\{\mathbf{b} \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\}$ is a splitting of M_1.

(2) $\Rightarrow$ (3). Suppose statement (2) holds for the connection matrix $\mathfrak{n}$ and let $\widehat{\mathfrak{n}}$ be any other connection matrix for $\mathcal{N}$. Then for each $j \in \mathbf{n}$, $\widehat{\mathfrak{n}^j} = \gamma^j \mathfrak{n}^j$ for some $\gamma^j \in G_{\mathcal{N}}$ ($\gamma^j \mathfrak{n}^j$ is the natural action of $G_{\mathcal{N}}$ on $\{j\} \times \prod_{i=1}^q \mathbf{n}^{s_i}$). For $j \in \mathbf{n}$, let $\mathbf{N}^j$ denote the set of j^{th} columns of all matrices in $\mathbf{N}$. Since $\{b \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\}$ is a splitting of M_1, $\{[\gamma^1(b) \star \mathbf{N}^1, \cdots, \gamma^n(b) \star \mathbf{N}^n] \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\} = \{[b \star \gamma^1(\mathbf{N}^1), \cdots, b \star \gamma^n(\mathbf{N}^n)] \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\}$ is a splitting of M_1. Hence (2) holds for $\widehat{\mathfrak{n}}$.

(3) $\Rightarrow$ (1). Take $L = [G_{\mathcal{N}} L^1, \cdots, G_{\mathcal{N}} L^u]$. Fix a connection matrix $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$ for $\mathcal{N}$ and denote the associated family of splittings of $N_1, \cdots, N_q$ by $\mathbf{N}$ as above. Since $\{\mathbf{b} \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\}$ is a splitting of M_1, we have a connection matrix $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ for $\mathcal{M}$, where $\mathfrak{m}^j = (\mathfrak{m}_1^j) \in \mathbf{n}^r$ satisfies

$$\mathbf{L}(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = \mathbf{X}_{\mathfrak{m}_1^j},\ j \in \mathbf{n}.$$

Hence for all $j \in \mathbf{n}$,

$$\begin{aligned} g(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) &= f(\mathbf{X}_j; \mathbf{L}(\mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j})) \\ &= f(\mathbf{X}_j; \mathbf{X}_{\mathfrak{m}_1^j}). \end{aligned}$$

This implies $\mathcal{M} \prec_I \mathcal{N}$. $\square$

Remark 5.2.13. If we have $\mathcal{M} \prec_I \mathcal{N}$ and we take the default connection matrix for $\mathcal{N}$, then the connection matrix on $\mathcal{M}$ given by Theorem 5.2.11(2) will generally *not* equal the default connection matrix of $\mathcal{M}$. For example, suppose that $\mathcal{N}$ is the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{M}$ is the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. We have $M_1 = N_0 + N_1$ and may easily check directly that $\mathcal{M} \prec_I \mathcal{N}$. If $\mathcal{F} \in \mathcal{M}$ has model $f : V \times V^2 \to V$, then we define g modelling $\mathcal{G} \in \mathcal{N}$ either by $g(x; y) = f(x; \overline{x, y})$ or by $g(x; y) = f(x; \overline{y, x})$. Here the only choices of L are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Neither of these choices gives the default connection matrix for $\mathcal{M}$.

Corollary 5.2.14. *(Notation and assumptions as above.) Suppose that* $M_1 \in M_1(n; \mathbb{Z}^+)$*, then* $\mathcal{M} \prec_I \mathcal{N}$ *iff* $M_1 \in \mathbf{A}(\mathcal{N})$.

Proof. ($\Rightarrow$): Since $\mathcal{M} \prec_I \mathcal{N}$, there is a linear transformation $\mathbf{L}$ with matrix $L = [\mathbf{a}] \in M(1, \sum_{j=0}^q s_j, \mathbb{Q})$, where $\mathbf{a} = [a_0; a_1, \cdots, a_q] \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$, such that $f \prec^\imath_{(\mathbf{L}, \mathfrak{m}, \mathfrak{n})} g$. Since L has only one row, $\mathcal{G}_\mathcal{N}\mathbf{a} = \{\mathbf{a}\}$. Therefore, for $j \in \mathbf{q}$, we may write $a_j = \lambda_j \mathbf{1} \in \mathbb{Q}^{s_j}$ where $\lambda_j \in \mathbb{Q}$. If we take $u = 1$, and $L^1 = \mathbf{a}$, then

$M_1 = \sum_{j=0}^{q} \lambda_j N_j$.

($\Leftarrow$): Let $M_1 = \sum_{j=0}^{q} \lambda_j N_j$. Take $\mathbf{L}$ to be the linear transformation with matrix $L = [\mathbf{a}] \in M(1, \sum_{j=0}^{q} s_j, \mathbb{Q})$, $\mathbf{a} = [a_0; a_1, \cdots, a_q] \in \mathbb{Q} \times \prod_{j=1}^{q} \mathbb{Q}^{s_j}$, where $a_j = \lambda_j \mathbf{1} \in \mathbb{Q}^{s_j}$, $j \in \mathbf{q}$. Hence we have

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = f(x_0; \sum_{j=0}^{q} \lambda_j \sum_{i=1}^{s_j} x_{ji}),$$

where $s_0 = 1$, $x_{01} = x_0$ and $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$ denotes variables corresponding to the inputs of type i, $i \in \mathbf{q}$. It is straightforward to check that $f \prec^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. $\square$

Corollary 5.2.15. *(Notation and assumptions as above.) If M_1 has a splitting $(Q_1, \cdots, Q_r)$ such that $\{Q_1, \cdots, Q_r\} \subseteq \mathbf{A}(\mathcal{N})$, then $\mathcal{M} \prec_I \mathcal{N}$.*

Proof. For each $i \in \mathbf{r}$, let $Q_i = \sum_{j=0}^{q} \lambda_{ij} N_j$. Define

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = f(x_0; \sum_{j=0}^{q} \lambda_{1j} \sum_{i=1}^{s_j} x_{ji}, \cdots, \sum_{j=0}^{q} \lambda_{rj} \sum_{i=1}^{s_j} x_{ji}),$$

where $s_0 = 1$, $x_{01} = x_0$ and $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$ denotes variables corresponding to the inputs of type i, $i \in \mathbf{q}$. It is straightforward to check that $f \prec^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. $\square$

Corollary 5.2.16. *(Notation and assumptions as above.) Suppose that $M_1 \in \mathbf{A}(\mathcal{N}, \mathbb{Z}^+)$, that is, $M_1 = \sum_{j=0}^{q} \alpha_j N_j$, $\alpha_j \in \mathbb{Z}^+$, $j \in \overline{\mathbf{q}}$. Then $\mathcal{M} \prec_I \mathcal{N}$.*

Proof. Define

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = f(x_0; x_0^{\alpha_0}; \mathbf{X}_1^{\alpha_1}, \cdots, \mathbf{X}_q^{\alpha_q}),$$

where $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$ denotes variables corresponding to the inputs of type i, $i \in \mathbf{q}$. It is straightforward to check that $f \prec^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. $\square$

Corollary 5.2.17. *(Notation and assumptions as above.) If we can write* $M_1 = A + S$ *where* $A \in \mathbf{A}(\mathcal{N}, \mathbb{Z}^+)$*, and there exists a splitting* $(S_1, \cdots, S_t)$ *of* S *such that* $S_i \in \mathbf{A}(\mathcal{N})$*,* $i \in \mathbf{t}$*, then* $\mathcal{M} \prec_I \mathcal{N}$*.*

Proof. Define the S component of M_1 using Corollary 5.2.15 and the A component using Corollary 5.2.16. □

Theorem 5.2.18. *(Notation and assumptions as above except that we allow* $p \geq 1$*.) The following statements are equivalent*

1. $\mathcal{M} \prec_I \mathcal{N}$*.*

2. *Suppose that* $\mathfrak{n}$ *is a connection matrix for* $\mathcal{N}$*. For* $j \in \mathbf{p}$*, there exist* $u_j \in \mathbb{N}$*,* $L_j^i \in \mathbb{Q} \times \prod_{v=1}^{q} \mathbb{Q}^{s_v}$*,* $i \in \mathbf{u_j}$*, such that* $\{\mathbf{b} \star \mathbf{N} \mid b \in G_{\mathcal{N}} L_j^i, i \in \mathbf{u_j}\}$ *is a splitting of* M_j*.*

Proof. The result is immediate from Theorem 5.2.11 and Lemma 5.2.3. □

Corollary 5.2.19. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be coupled* n *identical cell networks. Assume cells in* $\mathcal{M}$ *have* r *inputs, cells in* $\mathcal{N}$ *have* s *inputs. Suppose that* $\mathcal{M}$ *has adjacency matrices* $M_0 = I, M_1, \cdots, M_p$ *and* $\mathcal{N}$ *has adjacency matrices* $N_0 = I, N_1, \cdots, N_q$*. We assume that for each* $i \in \mathbf{p}$ *either* $r_i = 1$ *or* $s_j > r_i > 1$*, for all* $j \in \mathbf{q}$*. Under these conditions the following statements are equivalent*

1. $\mathcal{M} \prec_I \mathcal{N}$*.*

2. *For all* $i \in \mathbf{p}$*, there exists a splitting* $(P_{i,1}, \cdots, P_{i,r_i})$ *of* M_i *such that* $P_{i,j} \in \mathbf{A}(\mathcal{N})$*, for all* $j \in \mathbf{r}_i$*.*

Proof. (Sketch.) (2) $\Rightarrow$ (1) is trivial. In order to prove (1) $\Rightarrow$ (2), we may assume $p = 1$. Set $r = r_1$. For every $\mathbf{a} \in \mathbb{Q} \times \prod_{j=1}^{q} \mathbb{Q}^{s_j}$, $G_{\mathcal{N}}\mathbf{a}$ has one element or at least $\min_{j \in \mathbf{q}} s_j$ elements. Since $r < s_j$ for all $j \in \mathbf{q}$, we have $r < \min_{j \in \mathbf{q}} s_j$. Therefore L must be of the form $[L^1, \cdots, L^r]$ where $L^i \in \mathbb{Q} \times \prod_{j=1}^{q} \mathbb{Q}\mathbf{1}^{s_j}$. $\square$

Remark 5.2.20. If the network $\mathcal{M}$ has asymmetric inputs and $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$, hypothesis (2) of Corollary 5.2.19 is automatically satisfied (and so we recover the result for networks with asymmetric inputs — see Lemma 4.2.13 of Chapter 4. However, if $\mathcal{M}$ has symmetric inputs and $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$, then it need not be the case that (2) is satisfied (see Example 5.2.9, note that the only splitting of M_1 is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) and so $\mathcal{M}$ may not be input dominated by $\mathcal{N}$, even if we assume linear phase spaces or scaling signalling. We give some examples in the next section.

5.3 Examples

We conclude with two examples of network architectures that are both input and output equivalent as well as an example of self-output equivalence.

Example 5.3.1. If $p = q = 1$, $N_1 = bS$, and $M_1 = aS$ for $S \in M_1(n; \mathbb{Z}^+)$, $a, b \in \mathbb{N}$, then $\mathcal{M} \sim_O \mathcal{N}$ and $\mathcal{M} \sim_I \mathcal{N}$. Here $r = a$, $s = b$.

(a) Suppose $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^a \to TM$. Define $g : M \times M^b \to TM$ by

$$g_O(x_0; \overline{x_1, \cdots, x_b}) = \frac{1}{b}\left[f(x_0; x_1^a) + \cdots + f(x_0; x_b^a)\right],$$

$$g_I(x_0;\overline{x_1,\cdots,x_b}) \;=\; f(x_0;(\frac{1}{b}\sum_{i=1}^{b}x_i)^a),$$

where x^a signifies that x repeated a-times. It is easy to verify that g_O and g_I give the required output and input dominations of f. Hence, $\mathcal{M} \prec_O \mathcal{N}$ and $\mathcal{M} \prec_I \mathcal{N}$. The reverse order is obtained by interchanging a and b. Note that the input relations are same as were defined in Corollary 5.2.15.

Example 5.3.2. Let $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathcal{N}$ be the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Note that $N_1 = 2M_1$ and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M})$. We show that $\mathcal{M} \sim_O \mathcal{N}$ and $\mathcal{M} \sim_I \mathcal{N}$. (a) Suppose that $\mathcal{G} \in \mathcal{N}$ has model g. Let the system $\mathcal{F} \in \mathcal{M}$ have model

$$f(x_0;\overline{x_1,x_2}) = g(x_0;\overline{x_1,x_2,x_1,x_2}).$$

Then f output and input dominates g and so $\mathcal{N} \prec_I \mathcal{M}$ and $\mathcal{N} \prec_O \mathcal{M}$.

(b) Suppose that $\mathcal{F} \in \mathcal{M}$ has model f. Define g by

$$\begin{aligned} g_O(x_0;\overline{x_1,\cdots,x_4}) &= \frac{1}{4}\sum_{1\le i<j\le 4} f(x_0;\overline{x_i,x_j}) - \frac{1}{8}\sum_{1\le i\le 4} f(x_0;\overline{x_i,x_i}), \\ g_I(x_0;\overline{x_1,x_2,x_3,x_4}) &= f(x_0;x_0,\frac{x_1+x_2+x_3+x_4}{2}-x_0). \end{aligned}$$

Then g_O and g_I give the required output and input dominations of f and so $\mathcal{M} \prec_O \mathcal{N}$ and $\mathcal{M} \prec_I \mathcal{N}$.

5.4 Universal network

Definition 5.4.1. ([4]) Suppose that we consider the set $\mathrm{Net}(k)$ of networks with k identical cells. We can construct a maximal or *universal network* $\mathcal{U} \in \mathrm{Net}(k)$ for the order $\prec$. That is, for every $\mathcal{F} \in \mathcal{N} \in \mathrm{Net}(k)$, there exists $\mathcal{F}^\star \in \mathcal{U}$ such that $\mathcal{F}$ and $\mathcal{F}^\star$ have identical dynamics.

Remarks 5.4.2. (1) We assume that the state of the cells in the network depend on their internal state. Consequently, the identity matrix is one of the adjacency matrices.

(2) We denote the minimum number of inputs required to construct a universal network in $\mathrm{Net}(k)$ by $n(k)$.

(3) In general, even if $\mathcal{U}$ is universal and has the minimal number of inputs, $\mathcal{U}$ will not be unique (even up to isomorphism of associated graph structure). For example, in Figure 4.1, networks $\mathcal{M}$ and $\mathcal{N}$ are both universal in $\mathrm{Net}(2)$.

(4) By definition, all universal networks in $\mathrm{Net}(k)$ are equivalent.

Now we will find the precise value of $n(k)$ for each k. We divide the class $\mathrm{Net}(k)$ into five groups $\mathcal{N}_i(k) \subset \mathrm{Net}(k)$ and find the minimum number $n_i(k)$ of inputs required to construct a universal network in $\mathcal{N}_i(k)$, $1 \leq i \leq 6$.

1. $\mathcal{N}_1(k)$: Networks with all inputs of different types and no self loops.

2. $\mathcal{N}_2(k)$: Networks with all inputs of different types with self loops allowed.

3. $\mathcal{N}_3(k)$: Networks with different input types and no self loops.

4. $\mathcal{N}_4(k)$: Networks with different input types with self loops allowed.

5. $\mathcal{N}_5(k)$: Networks with all inputs of same type and no self loops.

6. $\mathcal{N}_6(k)$: Networks with all inputs of same type with self loops allowed.

Note that $\mathcal{N}_1(k), \mathcal{N}_5(k) \subset \mathcal{N}_3(k)$; $\mathcal{N}_2(k), \mathcal{N}_6(k) \subset \mathcal{N}_4(k)$.

Networks with all inputs of different types and no self loops

Let $A_{i_1,\cdots,i_k}$ denote the $k \times k$ $0-1$ matrix with unique non-zero entry 1 in row i_j of the jth column. Let $\Delta = \{(i_1, \cdots, i_k) \mid 1 \leq i_p \leq k, i_p \neq p, 1 \leq p \leq k\}$. The set $\mathcal{A} = \{A_{\mathbf{i}} \mid \mathbf{i} = (i_1, \cdots, i_k) \in \Delta\}$ is the set of all possible adjacency matrices, with $|\mathcal{A}| = (k-1)^k$. We want to find the number of linearly independent matrices in the set $\mathcal{A}$. Consider

$$B = \sum_{\mathbf{i} \in \Delta} a_{\mathbf{i}} A_{\mathbf{i}} = \mathbf{0}, \quad a_{\mathbf{i}} \in \mathbb{R}.$$

Thus we have $k(k-1)$ equations in $(k-1)^k$ unknowns. Hence, the set $\mathcal{A}$ has at most $k(k-1)$ linearly independent elements (matrices). Note that

$$\begin{aligned} B_{kj} &= \sum_{i=1}^{k} B_{i1} - \sum_{i=1}^{k-1} B_{ij}, \quad 2 \leq j \leq k, \\ B_{k(k-1)} &= \sum_{i} B_{i1} - \sum_{i=1}^{k-2} B_{ik}. \end{aligned}$$

Therefore, A has at most $(k-1)^2$ linearly independent elements (matrices). We show that A has exactly $(k-1)^2$ linearly independent elements (matrices). The equations are: $B_{i1} = 0, 2 \leq i \leq k$, $B_{pq} = 0, 1 \leq p \leq k-1, 2 \leq q \leq k, p \neq q$. Define a mapping $\phi : \Delta \to \{1, 2, \cdots, (k-1)^k\}$. Let $\mathbf{a} = (a_{\phi^{-1}(1)}, \cdots, a_{\phi^{-1}((k-1)^k)})^t$,

$B_{ij} = P_{ij}\mathbf{a}$, where $P_{ij} \in \mathbb{R}^{(k-1)^k}$. Let

$$\sum_{ij} t_{ij} P_{ij} = 0, \;\; t_{ij} \in \mathbb{R}.$$

The entry at $\phi((i_1, k, \cdots, k))$ place is $t_{i_1 1}$, and must be equal to 0, for all $i_1 = 2, \cdots, k$. The entry at $\phi((i_1, i_2, k, \cdots, k))$ place is $t_{i_2 2}$, and must be equal to 0, for all $i_2 = 1, 3, \cdots, k$. Similarly, observing the entries at $\phi((i_1, i_2, i_3, k, \cdots, k))$,$\cdots$, $\phi(i_1, \cdots, i_{k-1}, k)$ place, we get $t_{ij} = 0$, for all i, j. Therefore, there are $(k-1)^2$ linearly independent matrices in the set $\mathcal{A}$. Thus, $n_1(k) = (k-1)^2$.

Networks with all inputs of different types and self loops allowed

Let $B_{i_1, \cdots, i_k}$ denote the $k \times k$ $0-1$ matrix with unique non-zero entry 1 in row i_j of the jth column. Let $\Delta' = \{(i_1, \cdots, i_k) \mid 1 \leq i_p \leq k, 1 \leq p \leq k\}$. The set $\mathcal{B} = \{B_{\mathbf{i}} \mid \mathbf{i} = (i_1, \cdots, i_k) \in \Delta'\}$ is the set of all possible adjacency matrices. A similar proof, to that given above, shows that there are $k^2 - k + 1$ linearly independent matrices in $\mathcal{B}$. Since the identity matrix is included in the set $\mathcal{B}$, so if the state of the cells in the network depends on the internal state, then $k^2 - k$ inputs are required to form a universal network. Thus, $n_2(k) = k^2 - k$.

Networks with different input types and no self loops

For a general network with identical cells where we allow symmetric inputs, an adjacency matrix is of the form $M = [m_{ij}]$. Since the cells are identical, the

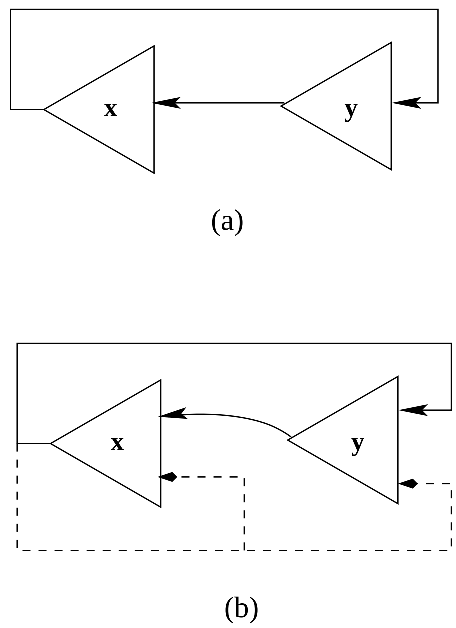
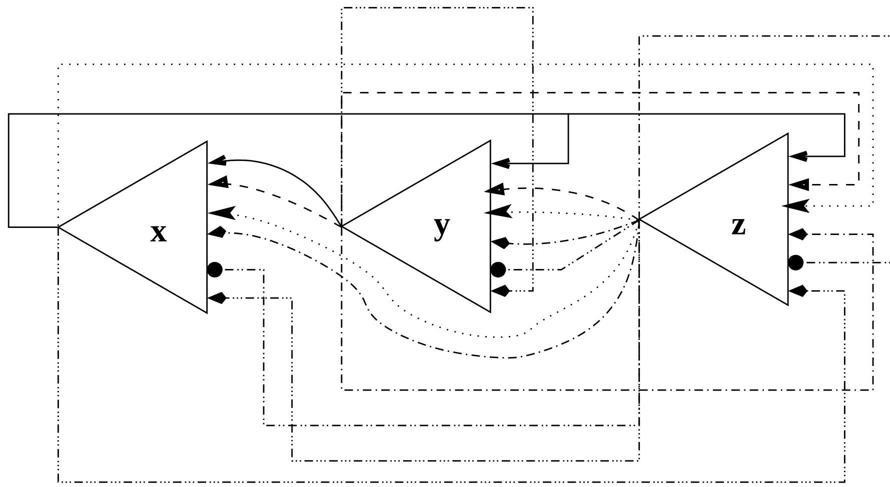


Figure 5.3: Universal network of 2 cells - (a) without self loops, (b) with self loops allowed.

column sum of M is constant. Therefore, M can be written as a sum of matrices in the set $\mathcal{A}$. Therefore, $\mathcal{A}$ spans the set of all possible adjacency matrices for networks in $\mathcal{N}_3(k)$. Hence, $n_3(k) = (k-1)^2$.

Networks with different input types and self loops allowed

A similar argument shows that $n_4(k) = k^2 - k$.



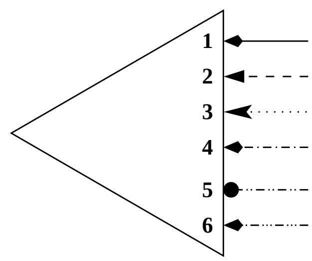


Figure 5.4: Universal network of 3 cells. The network with edges labelled $1, \cdots, 4$ is a universal network without self loops; together with edges labelled $5, 6$ is a universal network with self loops allowed.

Networks with all inputs of same type (or, homogeneous networks)

(1) **No self loops**: Let $\mathcal{U} \in \mathcal{N}_5$ be a universal network with adjacency matrix $M = [m_{ij}]$. Note that $\mathcal{U}$ is unique in $\mathcal{N}_5(k)$. Let

$$d = \gcd\{m_{ij} \mid j \neq i\}.$$

Since $\mathcal{U}$ has minimum number of inputs and has no self loops, $d = 1$ and $m_{ii} = 0$, for all $i = 1, \cdots, k$. Since, $\mathcal{U}$ is universal, for every $\mathcal{F} \in \mathcal{N} \in \mathcal{N}_5(k)$ with adjacency matrix N, N is a non-zero multiple of M. This is not true; there exist networks in $\mathcal{N}_5(k)$ whose adjacency matrices are not multiples of M. Hence, there is no universal network in $\mathcal{N}_5(k)$.

(2) **Self loops allowed**: Let $\mathcal{U} \in \mathcal{N}_6(k)$ be a universal network with adjacency matrix $M = [m_{ij}]$. Note that $\mathcal{U}$ is unique in $\mathcal{N}_6(k)$. Let

$$d = \gcd\{m_{ij} \mid j \neq i\}, \quad a = \min\{m_{ii} \mid i = 1, \cdots, k\}.$$

Since $\mathcal{U}$ has minimum number of inputs, $d = 1$ and $a = 0$ (See [6, Theorem 10.3]). Let $m_{i_0 i_0} = 0$. Since, $\mathcal{U}$ is universal, for every $\mathcal{F} \in \mathcal{N} \in \mathcal{N}_6(k)$ with adjacency matrix N, $N = pI + qM$ for some $p, q \in \mathbb{R}$. Consider a network $\mathcal{N} \in \mathcal{N}_6(k)$ in which just the i_0^{th} cell has a self loop then the adjaceny matrix N of $\mathcal{N}$, N can not be written as $N = pI + qM$ for some $p, q \in \mathbb{R}$. Hence, there is no universal network in $\mathcal{N}_6(k)$.

Remark 5.4.3. Networks in $\mathcal{N}_5(k)$ and $\mathcal{N}_6(k)$ have universal networks in $\mathcal{N}_3(k)$ and $\mathcal{N}_6(k)$, respectively.

Chapter 6

Inflation of Strongly Connected Networks

Inflation is the process of naturally embedding a smaller network into larger network so that the dynamics of the smaller network can be realized in the dynamics of the larger network restriced to an invariant subspace or synchrony subspace. Inflation can be viewed as inverse to the process of forming the *quotient network* as defined by Stewart *et al.* [23],[25, Chapter 9]. Inflation can also be used to construct and identify networks that support, for example, heteroclinic cycles or heteroclinic switching networks (we refer to [4] for more details, examples and background). The main goal of this chapter is to provide a simple necessary and sufficient condition for the existence of a strongly connected inflation.

Associated to every coupled cell network, there is an underlying directed graph or *digraph* whose vertices are cells and the directed edges are the connections in the network. A network is *strongly connected* or *transitive* if the associated graph is strongly connected in the sense of graph theory; that is, between any ordered pair of nodes, there is a path of connections joining the first to the second node. In this chapter we restrict to strongly connected networks. It is well known that strong connectivity plays an important role in understanding the dynamics of coupled cell systems. For example, it is proved in [49] that synchronization of all the cells is possible in a strongly connected coupled cell network if the coupling strength is sufficiently large. Timme [47] has observed that the partition of a network into strongly connected components is important in understanding network dynamics and synchronization. In particular, cells tend to synchronize faster within strongly connected components.

The inflation of an undirected graph is also defined in graph theory (for example [22, 41]) but the definition of inflation we use here is different, is specific to directed graphs and was introduced in [4] in the context of coupled cell networks.

A coupled cell network can be viewed as a graph where vertices correspond to the cells, edges to connections. We regard two cells as being of same *class* if the same inputs result in same output. In diagrams, we will typically follow the linear systems representation of Figure 2.8(b) and use triangles for representing cells. We refer to Chapter 2 for general background on this approach to coupled cell networks.

Definition 6.0.4 ([4]). Let $\mathcal{M}$ be a coupled cell network with cell set $\mathcal{C} = \{C_1, \cdots, C_k\}$. A coupled cell network $\mathcal{N}$ is an *inflation* of $\mathcal{M}$ if there exists a surjective morphism (of graphs) $\Pi : \mathcal{N} \to \mathcal{M}$ sending cells to cells (preserving class) and connections to connections (preserving type) such that

(1) $\{\Pi^{-1}(C_1)\| \cdots \|\Pi^{-1}(C_k)\}$ is a synchrony subspace of $\mathcal{N}$.

(2) Π maps the set of connections in $\mathcal{N}$ onto the set of connections in $\mathcal{M}$. More precisely, there is a connection of type ℓ from C_i to C_j, if and only if there exist cells $C_{i\alpha} \in \Pi^{-1}(C_i)$ and $C_{j\beta} \in \Pi^{-1}(C_j)$ such that there is a connection of type ℓ from $C_{i\alpha}$ to $C_{j\beta}$.

Definition 6.0.5. (unidirectional ring) Let $\mathcal{M}$ be a coupled cell network with cell set $\{C_1, \cdots, C_n\}$ and p inputs (symmetric or asymmetric). We say that $\mathcal{M}$ is a *p-input unidirectional ring* if there is an ordering of cells $C_{i_1}, \cdots, C_{i_n}$ such that C_{i_k} has all the inputs from $C_{i_{k-1}}$, $2 \leq k \leq n$ and C_{i_1} has all the inputs from C_{i_n}. In other words, there is a relabelling of cells such that the adjacency matrix A takes the form

$$\begin{pmatrix} \mathbf{0}_{n-1,1} & p\mathbf{I}_{n-1} \\ p & \mathbf{0}_{1,n-1} \end{pmatrix}.$$

Lemma 6.0.6. *Suppose that* $\Pi : \mathcal{N} \to \mathcal{M}$ *defines an inflation* $\mathcal{N}$ *of the network* $\mathcal{M}$. *If* $\{S_1 \| \cdots \| S_p\}$ *is a synchrony subspace of* $\mathcal{M}$, *then* $\{\Pi^{-1}(S_1) \| \cdots \| \Pi^{-1}(S_p)\}$ *is a synchrony subspace of* $\mathcal{N}$.

Proof. The proof amounts to showing that if $i \in \mathbf{p}$ and D_α, D_β are cells in $\Pi^{-1}(S_i)$, then for all $j \in \mathbf{p}$, D_α, D_β see the same set of inputs from cells in $\Pi^{-1}(S_j)$. This is an immediate consequence of part (2) of Definition 6.0.4 together with our assumption that $\{S_1 \| \cdots \| S_p\}$ is a synchrony subspace. $\square$

Definition 6.0.7 ([4]). A coupled cell network $\mathcal{N}$ is a *p-fold simple inflation* of $\mathcal{M}$ at C_i if

(1) the cell set of $\mathcal{N}$ is given by $(\mathcal{C} \setminus C_i) \cup \{C_{i1}, \cdots, C_{ip}\}$, where $\{C_{i1}, \cdots, C_{ip}\}$ are of same cell class as C_i.

(2) $\{C_{i1}, \cdots, C_{ip}\} = \{C_1\| \cdots \|C_{i1}, \cdots, C_{ip}\| \cdots \|C_k\}$ is a synchrony class of $\mathcal{N}$ and the induced network structure on the synchrony subspace $\{C_{i1}, \cdots, C_{ip}\}$ is equal to that of $\mathcal{M}$.

Remark 6.0.8. If $\Pi : \mathcal{N} \to \mathcal{M}$ is an inflation of the network $\mathcal{M}$, then $\mathcal{N}$ is a simple inflation of $\mathcal{M}$ iff there exists $i \in \mathbf{k}$ such that $\Pi^{-1}(C_i) = \{C_{i1}, \cdots, C_{ip}\}$ and $\Pi^{-1}(C_j) = \{C_j\}$, $j \neq i$.

A strongly connected network which has no self inputs and with every cell having exactly one output is (after relabelling of cells) a one-input unidirectional ring $C_1 \to C_2 \to \cdots \to C_k \to C_1$ and has no simple inflations. A network $\mathcal{M}$ will have a *p-fold* simple inflation at C_i iff either C_i has a self loop or C_i has at least $p > 1$ outputs. If $\mathcal{N}$ is a simple inflation of $\mathcal{M}$, we write $\mathcal{M} \Longleftarrow_S \mathcal{N}$. In Figure 6.1, we give diagrammatic illustration of the process of simple inflation. In the figure, $\mathcal{N}$ is a *2-fold* simple inflation of $\mathcal{M}$ at C_2 (C_2 has two outputs). Other properties and examples of inflation are given in [4].

Definition 6.0.9. Let $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k) \in \mathbb{N}^k$. If $\Pi : \mathcal{N} \to \mathcal{M}$ is an inflation of $\mathcal{M}$ such that $\#\Pi^{-1}(C_i) = n_i$, $i \in \mathbf{k}$, then $\mathcal{N}$ is said to be an $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{M}$.

Remark 6.0.10. $\overrightarrow{\mathbf{n}}$-inflation of a network need not be unique. In Example 6.0.11, $\mathcal{M}_i$, $i \in \bar{\mathbf{9}}$ are distinct $(2, 2)$-inflations of the network $\mathcal{M}$.

We will now look at the form of the adjacency matrix of the inflated network. Let $\mathcal{N}$ be an inflation of a given network $\mathcal{M}$ with adjacency matrix $A = [a_{ij}]$. The

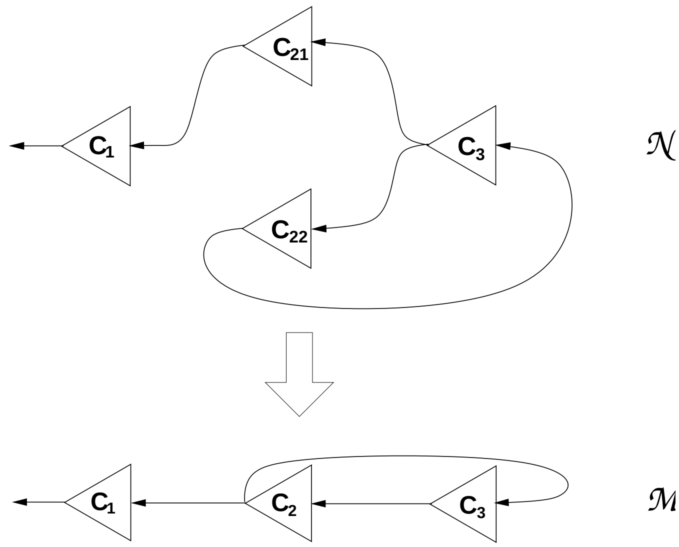


Figure 6.1: $\mathcal{N}$ is a *2-fold* simple inflation of $\mathcal{M}$.

adjacency matrix of $\mathcal{N}$ is

$$\overline{A} = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ & \ddots & \\ A_{k1} & \cdots & A_{kk} \end{pmatrix},$$

where A_{ij} is a $n_i \times n_j$ matrix, the columns are enumerated as n_j inflated cells of C_j and the rows are enumerated as n_i inflated cells of C_i. By the definition of inflation and Remark 2.3.1, the column sum of each A_{ij} is constant, and is equal to a_{ij}. This leads to the result of Aguiar *et al.* [7, §2]. Thus, $\mathcal{N}$ is an inflation of $\mathcal{M}$ if and only if $\mathcal{M}$ is a quotient of $\mathcal{N}$.

Table 6.1: Inequivalent $(2,2)$-inflations of $\mathcal{M}$ (symmetric inputs).

	System	Non-trivial synchrony classes	Comments
$\mathcal{M}$	$x_1' = f(x_1; x_2, x_2)$ $x_2' = f(x_2; x_1, x_1)$		two-input unidirectional ring $C_1 \to C_2 \to C_1$
$\mathcal{M}_0$	$x_{11}' = f(x_{11}; x_{21}, x_{21})$ $x_{12}' = f(x_{12}; x_{22}, x_{22})$ $x_{21}' = f(x_{21}; x_{11}, x_{11})$ $x_{22}' = f(x_{22}; x_{12}, x_{12})$	$\{x_{11}, x_{12} \Vert x_{21}, x_{22}\}$ $\{x_{11}, x_{21} \Vert x_{12}, x_{22}\}$ $\{x_{12}, x_{22}\}$ $\{x_{11}, x_{21}\}$	not connected
$\mathcal{M}_1$	$x_{11}' = f(x_{11}; x_{21}, x_{22})$ $x_{12}' = f(x_{12}; x_{21}, x_{22})$ $x_{21}' = f(x_{21}; x_{11}, x_{11})$ $x_{22}' = f(x_{22}; x_{12}, x_{12})$	$\{x_{11}, x_{12} \Vert x_{21}, x_{22}\}$ $\{x_{11}, x_{12}\}$ $\{x_{21}, x_{22}\}$	
$\mathcal{M}_2$	$x_{11}' = f(x_{11}; x_{21}, x_{22})$ $x_{12}' = f(x_{12}; x_{21}, x_{21})$ $x_{21}' = f(x_{21}; x_{11}, x_{12})$ $x_{22}' = f(x_{22}; x_{11}, x_{12})$	$\{x_{11}, x_{12} \Vert x_{21}, x_{22}\}$ $\{x_{21}, x_{22}\}$	
$\mathcal{M}_3$	$x_{11}' = f(x_{11}; x_{21}, x_{22})$ $x_{12}' = f(x_{12}; x_{21}, x_{21})$ $x_{21}' = f(x_{21}; x_{11}, x_{11})$ $x_{22}' = f(x_{22}; x_{12}, x_{12})$	$\{x_{11}, x_{12} \Vert x_{21}, x_{22}\}$	
$\mathcal{M}_4$	$x_{11}' = f(x_{11}; x_{21}, x_{22})$ $x_{12}' = f(x_{12}; x_{21}, x_{22})$ $x_{21}' = f(x_{21}; x_{11}, x_{12})$ $x_{22}' = f(x_{22}; x_{11}, x_{12})$	$\{x_{11}, x_{12} \Vert x_{21}, x_{22}\}$ $\{x_{11}, x_{22} \Vert x_{12}, x_{21}\}$ $\{x_{11}, x_{21} \Vert x_{12}, x_{22}\}$ $\{x_{11}, x_{12}\}$ $\{x_{21}, x_{22}\}$	the synchrony class $\{x_{11}, x_{22} \Vert x_{12}, x_{21}\}$ depends on symmetry of inputs
$\mathcal{M}_5$	$x_{11}' = f(x_{11}; x_{22}, x_{22})$ $x_{12}' = f(x_{12}; x_{21}, x_{21})$ $x_{21}' = f(x_{21}; x_{11}, x_{11})$ $x_{22}' = f(x_{22}; x_{12}, x_{12})$	$\{x_{11}, x_{12} \Vert x_{21}, x_{22}\}$	two-input unidirectional ring $C_{11} \to C_{21} \to C_{12} \to C_{22} \to C_{11}$
$\mathcal{M}_6$	$x_{11}' = f(x_{11}; x_{21}, x_{22})$ $x_{12}' = f(x_{12}; x_{21}, x_{21})$ $x_{21}' = f(x_{21}; x_{12}, x_{12})$ $x_{22}' = f(x_{22}; x_{11}, x_{11})$	$\{x_{11}, x_{12} \Vert x_{21}, x_{22}\}$ $\{x_{12}, x_{21}\}$	

Continued on next page

Table 6.1 – continued from previous page

	System	Non-trivial synchrony classes	Comments
$\mathcal{M}_7$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$ $x'_{12} = f(x_{12}; x_{21}, x_{21})$ $x'_{21} = f(x_{21}; x_{11}, x_{11})$ $x'_{22} = f(x_{22}; x_{11}, x_{12})$	$\{x_{11}, x_{12} \| x_{21}, x_{22}\}$	
$\mathcal{M}_8$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$ $x'_{12} = f(x_{12}; x_{21}, x_{21})$ $x'_{21} = f(x_{21}; x_{11}, x_{12})$ $x'_{22} = f(x_{22}; x_{11}, x_{11})$	$\{x_{11}, x_{12} \| x_{21}, x_{22}\}$ $\{x_{11}, x_{21} \| x_{12}, x_{22}\}$	
$\mathcal{M}_9$	$x'_{11} = f(x_{11}; x_{21}, x_{21})$ $x'_{12} = f(x_{12}; x_{21}, x_{22})$ $x'_{21} = f(x_{21}; x_{11}, x_{11})$ $x'_{22} = f(x_{22}; x_{11}, x_{12})$	$\{x_{11}, x_{12} \| x_{21}, x_{22}\}$ $\{x_{11}, x_{21} \| x_{12}, x_{22}\}$	

Example 6.0.11. Let $\mathcal{M}$ be a two-input unidirectional ring (symmetric inputs) with adjacency matrix

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

In this case, $\mathcal{M}$ is a two-input unidirectional ring. Let $\mathcal{N}$ be a $(2,2)$-inflation of $\mathcal{M}$ with adjacency matrix $\overline{A}$ of the form described above. Then A_{11} and A_{22} are square zero matrices of size 2, A_{12} and A_{21} take either of the following forms

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Using [18] and Chapter 5, it can be shown that $\mathcal{M}$ has exactly 10 dynamically inequivalent $(2,2)$-inflations, $\mathcal{M}_i$, $i \in \bar{\mathbf{9}}$ (we refer to Chapter 4 and [4, 18] for *dynamical equivalence* which is an invariant of network architecture. Our example is used for illustrative purposes and we do not make any use of dynamical equivalence in the statement and formulation of our main results). We show the network architectures of these inflations in Figure 6.2.

We can clearly see that $\mathcal{M}_i$, $i \in \bar{\mathbf{4}}$ are all dynamically inequivalent since the number of synchrony classes is different for all of them (dynamically equivalent networks have same number of synchrony classes). The only inflation that is not strongly connected is $\mathcal{M}_0$ which consists of two copies of the original network $\mathcal{M}$. There are no non-trivial synchrony classes of the network $\mathcal{M}$ and so $\{x_{11}, x_{12} \| x_{21}, x_{22}\}$ is the only non-trivial synchrony class of the strongly connected networks $\mathcal{M}_i$, $i \in \bar{\mathbf{9}}$, that lifts from $\mathcal{M}$. We remark that the constructions that we describe later (in Theorem 6.1.7) only generate the inflations $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_4$.

Table 6.2: Inequivalent $(2,2)$-inflations of $\mathcal{M}$ (asymmetric inputs).

System				**Non-trivial synchrony classes**
$\mathcal{M}$	$x'_1 = f(x_1; x_2, x_2)$			
	$x'_2 = f(x_2; x_1, x_1)$			
$\mathcal{M}_{01}$	$x'_{11} = f(x_{11}; x_{21}, x_{21})$			$\{x_{11}, x_{21} \| x_{12}, x_{22}\}$
	$x'_{12} = f(x_{12}; x_{22}, x_{22})$			$\{x_{11}, x_{21}\}$
	$x'_{21} = f(x_{21}; x_{11}, x_{11})$			$\{x_{12}, x_{22}\}$
	$x'_{22} = f(x_{22}; x_{12}, x_{12})$			
$\mathcal{M}_{11}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{12}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{11} : \{x_{11}, x_{12}\}$
	$x'_{12} = f(x_{12}; x_{21}, x_{22})$		$x'_{12} = f(x_{12}; x_{22}, x_{21})$	$\{x_{21}, x_{22}\}$
	$x'_{21} = f(x_{21}; x_{11}, x_{11})$		$x'_{21} = f(x_{21}; x_{11}, x_{11})$	

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Table 6.2 – continued from previous page

System				Non-trivial synchrony classes
	$x'_{22} = f(x_{22}; x_{12}, x_{12})$		$x'_{22} = f(x_{22}; x_{12}, x_{12})$	
$\mathcal{M}_{21}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{22}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{21} : \{x_{11}, x_{21} \| x_{12}, x_{22}\}$
	$x'_{12} = f(x_{12}; x_{21}, x_{21})$		$x'_{12} = f(x_{12}; x_{21}, x_{21})$	$\{x_{21}, x_{22}\}$
	$x'_{21} = f(x_{21}; x_{11}, x_{12})$		$x'_{21} = f(x_{21}; x_{11}, x_{12})$	$\mathcal{M}_{22} : \{x_{11}, x_{21} \| x_{12}, x_{22}\}$
	$x'_{22} = f(x_{22}; x_{11}, x_{12})$		$x'_{22} = f(x_{22}; x_{12}, x_{11})$	
$\mathcal{M}_{31}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$			
	$x'_{12} = f(x_{12}; x_{21}, x_{21})$			
	$x'_{21} = f(x_{21}; x_{11}, x_{11})$			
	$x'_{22} = f(x_{22}; x_{12}, x_{12})$			
$\mathcal{M}_{41}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{42}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{41} : \{x_{11}, x_{21} \| x_{12}, x_{22}\}$
	$x'_{12} = f(x_{12}; x_{22}, x_{21})$		$x'_{12} = f(x_{12}; x_{21}, x_{22})$	$\{x_{11}, x_{22} \| x_{12}, x_{21}\}$
	$x'_{21} = f(x_{21}; x_{11}, x_{12})$		$x'_{21} = f(x_{21}; x_{11}, x_{12})$	$\mathcal{M}_{42} : \{x_{11}, x_{21} \| x_{12}, x_{22}\}$
	$x'_{22} = f(x_{22}; x_{12}, x_{11})$		$x'_{22} = f(x_{22}; x_{11}, x_{12})$	$\{x_{11}, x_{12}\}$
$\mathcal{M}_{43}$	$x'_{11} = f(x_{11}; x_{22}, x_{21})$	$\mathcal{M}_{44}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{44} : \{x_{11}, x_{21} \| x_{12}, x_{22}\}$
	$x'_{12} = f(x_{12}; x_{21}, x_{22})$		$x'_{12} = f(x_{12}; x_{21}, x_{22})$	$\{x_{11}, x_{12}\}$
	$x'_{21} = f(x_{21}; x_{11}, x_{12})$		$x'_{21} = f(x_{21}; x_{11}, x_{12})$	
	$x'_{22} = f(x_{22}; x_{12}, x_{11})$		$x'_{22} = f(x_{22}; x_{12}, x_{11})$	
$\mathcal{M}_{51}$	$x'_{11} = f(x_{11}; x_{21}, x_{21})$			
	$x'_{12} = f(x_{12}; x_{22}, x_{22})$			
	$x'_{21} = f(x_{21}; x_{11}, x_{11})$			
	$x'_{22} = f(x_{22}; x_{12}, x_{12})$			
$\mathcal{M}_{61}$	$x'_{11} = f(x_{11}; x_{21}, x_{21})$			$\{x_{12}, x_{21}\}$
	$x'_{12} = f(x_{12}; x_{21}, x_{21})$			
	$x'_{21} = f(x_{21}; x_{12}, x_{12})$			
	$x'_{22} = f(x_{22}; x_{11}, x_{11})$			
$\mathcal{M}_{71}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{72}$	$x'_{11} = f(x_{11}; x_{22}, x_{21})$	
	$x'_{12} = f(x_{12}; x_{21}, x_{21})$		$x'_{12} = f(x_{12}; x_{21}, x_{21})$	
	$x'_{21} = f(x_{21}; x_{11}, x_{11})$		$x'_{21} = f(x_{21}; x_{11}, x_{11})$	
	$x'_{22} = f(x_{22}; x_{11}, x_{12})$		$x'_{22} = f(x_{22}; x_{11}, x_{12})$	
$\mathcal{M}_{81}$	$x'_{11} = f(x_{11}; x_{21}, x_{22})$	$\mathcal{M}_{82}$	$x'_{11} = f(x_{11}; x_{22}, x_{21})$	$\mathcal{M}_{81} : \{x_{11}, x_{21} \| x_{12}, x_{22}\}$
	$x'_{12} = f(x_{12}; x_{21}, x_{21})$		$x'_{12} = f(x_{12}; x_{21}, x_{21})$	
	$x'_{21} = f(x_{21}; x_{11}, x_{12})$		$x'_{21} = f(x_{21}; x_{11}, x_{12})$	
	$x'_{22} = f(x_{22}; x_{11}, x_{11})$		$x'_{22} = f(x_{22}; x_{11}, x_{11})$	

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Table 6.2 – continued from previous page

System				Non-trivial synchrony classes
$\mathcal{M}_{91}$	$x'_{11} = f(x_{11}; x_{21}, x_{21})$	$\mathcal{M}_{92}$	$x'_{11} = f(x_{11}; x_{21}, x_{21})$	$\mathcal{M}_{91} : \{x_{11}, x_{21} \Vert x_{12}, x_{22}\}$
	$x'_{12} = f(x_{12}; x_{21}, x_{22})$		$x'_{12} = f(x_{12}; x_{22}, x_{21})$	
	$x'_{21} = f(x_{21}; x_{11}, x_{11})$		$x'_{21} = f(x_{21}; x_{11}, x_{11})$	
	$x'_{22} = f(x_{22}; x_{11}, x_{12})$		$x'_{22} = f(x_{22}; x_{11}, x_{12})$	

Example 6.0.12. Let $\mathcal{M}$ be a two-input unidirectional ring (asymmetric inputs) with both edge type adjacency matrices equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The network $\mathcal{M}$ has exactly 18 inequivalent $(2,2)$-inflations. We list these in Table 6.2. Note that $\mathcal{M}_{ij}$ is obtained from $\mathcal{M}_i$ in Table 6.1 by labelling the inputs (so that they are asymmetric). The non-trivial synchrony classes of $\mathcal{M}_{ij}$, other than $\{x_{11}, x_{12} \Vert x_{21}, x_{22}\}$, are listed in the last column of Table 6.2.

The composition of simple inflations need not be a simple inflation (for example, if different cells are inflated). However, composition of inflations is an inflation.

Lemma 6.0.13. *If $\mathcal{N}$ is an inflation of $\mathcal{M}$ and $\mathcal{K}$ is an inflation of $\mathcal{N}$ then $\mathcal{K}$ is an inflation of $\mathcal{M}$.*

Proof. Let $\Pi_1 : \mathcal{N} \to \mathcal{M}$, $\Pi_2 : \mathcal{K} \to \mathcal{N}$ define inflations $\mathcal{N}$ and $\mathbb{K}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively. It suffices to show that the composition $\Pi = \Pi_1 \circ \Pi_2 : \mathcal{K} \to \mathcal{M}$ is an inflation. We start by verifying that Π satisfies Definition 6.0.4(2). Let $C_1, \cdots, C_k$ denote the cells of $\mathcal{M}$. There is a connection of type ℓ from C_i to C_j, if and only if there exist cells $C_{i\alpha} \in \Pi_1^{-1}(C_i)$ and $C_{j\beta} \in \Pi_1^{-1}(C_j)$ such that there is a connection of type ℓ from $C_{i\alpha}$ to $C_{j\beta}$. Also, there is a connection of type ℓ from $C_{i\alpha}$ to $C_{j\beta}$, if

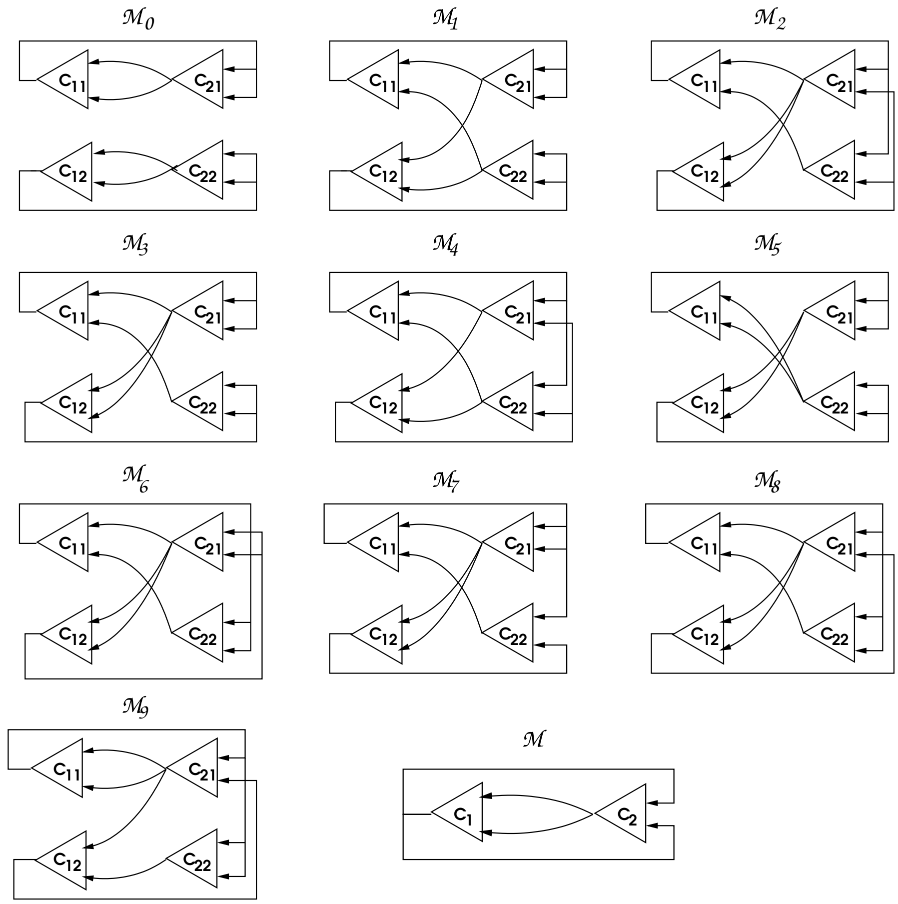

Figure 6.2: Network architectures of $\mathcal{M}_0, \cdots, \mathcal{M}_9$.

and only if there exist cells $C_{i\alpha r} \in \Pi_2^{-1}(C_{i\alpha})$ and $C_{j\beta s} \in \Pi_2^{-1}(C_{j\beta})$ such that there is a connection of type ℓ from $C_{i\alpha r}$ to $C_{j\beta s}$. Thus, there is a connection of type ℓ from C_i to C_j, if and only if there exist cells $C_{i\alpha r} \in \Pi^{-1}(C_i)$ and $C_{j\beta s} \in \Pi^{-1}(C_j)$ such that there is a connection of type ℓ from $C_{i\alpha r}$ to $C_{j\beta s}$. Finally, we must show that $\{\Pi^{-1}(C_1)\| \cdots \|\Pi^{-1}(C_k)\}$ is a synchrony subspace of $\mathcal{K}$. This follows by Lemma 6.0.6 applied to Π_2 since $\{\Pi_1^{-1}(C_1)\| \cdots \|\Pi_1^{-1}(C_k)\}$ is a synchrony subspace of $\mathcal{N}$. $\square$

Remark 6.0.14. Let $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_p) \in \mathbb{N}^p$. Set $q = \sum_{i=1}^{p} n_i$ and let

$$\overrightarrow{\mathbf{m}} = (m_{11}, \cdots, m_{1n_1}, \cdots, m_{pn_1}, \cdots, m_{pn_p}) \in \mathbb{N}^q.$$

Label the cells of $\mathcal{M}$ by $C_1, \cdots, C_p$. For $i \in \mathbf{p}$, let $r_i = \sum_{j=1}^{n_i} m_{ij}$. Set $\overrightarrow{\mathbf{r}} = (r_1, \cdots, r_p) \in \mathbb{N}^p$. Suppose $\mathcal{N}$ is an $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{M}$. Label the cells of $\mathcal{N}$ by

$$C_{11}, \cdots, C_{1n_1}, \cdots, C_{pn_1}, \cdots, C_{pn_p},$$

where $C_{i1}, \cdots, C_{in_i}$ are obtained by inflating C_i, for $i \in \mathbf{p}$. Suppose $\mathcal{K}$ is an $\overrightarrow{\mathbf{m}}$-inflation of $\mathcal{N}$, where m_{ij} is the number of cells by which C_{ij} is inflated, for $j \in \mathbf{n_p}$, $i \in \mathbf{p}$. Then $\mathcal{K}$ is an $\overrightarrow{\mathbf{r}}$-inflation of $\mathcal{M}$.

Let $\mathcal{M}$ be a coupled cell network and S be a synchrony subspace of $\mathcal{M}$. The network obtained by restricting $\mathcal{M}$ to S is called the *quotient network* of $\mathcal{M}$. If $\mathcal{N}$ is a quotient of $\mathcal{M}$ then $\mathcal{M}$ is an inflation of $\mathcal{N}$ defined by $\Pi : \mathcal{M} \to \mathcal{N}$. In Example 6.0.11, $\mathcal{M}_0, \cdots, \mathcal{M}_9$ are inflations of $\mathcal{M}$ and $S = \{C_{11}, C_{12} \| C_{21}, C_{22}\}$ is a synchrony subspace of $\mathcal{M}_i$, $i \in \bar{\mathbf{9}}$. Each network $\mathcal{M}_i$ restricted to S yields $\mathcal{M}$ (identify C_{11}, C_{12}, and C_{21}, C_{22}). Hence $\mathcal{M}$ is the quotient network of $\mathcal{M}_i$, $i \in \bar{\mathbf{9}}$ (see also [49, 7, 4, 20]).

6.1 Networks with a single input type

For the remainder of the paper, we assume all networks are strongly connected and we only consider strongly connected inflations of strongly connected networks. Typically we use the term 'inflation' to mean 'strongly connected inflation'. In this section we consider networks with only one type of connection (inputs are necessarily symmetric).

6.1.1 Networks with no self loops

We first consider networks with no self loops.

Definition 6.1.1. Let $\mathcal{M}$ be a coupled cell network with cell set $\mathcal{C}$ and $C, D \in \mathcal{C}$. The cell C is said to be *path connected* to D, written $C \rightsquigarrow_p^{\mathcal{M}} D$, if there is a path from C to D.

Definition 6.1.2. Let $\mathcal{M}$ be a coupled cell network with cell set $\mathcal{C}$ and suppose $A \subseteq \mathcal{C}$. If $C, D \in A$, then C is said to be *path connected in A to D*, written $C \rightsquigarrow_{p,A}^{\mathcal{M}} D$, if there is a path from C to D containing only cells from the set A.

Remark 6.1.3. Both relations $\rightsquigarrow_p^{\mathcal{M}}$ and $\rightsquigarrow_{p,A}^{\mathcal{M}}$ are obviously transitive.

Lemma 6.1.4. *Let $\mathcal{M}$ be a coupled cell network with k cells, labelled $C_1, \cdots, C_k$. Let A be the adjacency matrix of $\mathcal{M}$. If $\mathcal{N}$ is a strongly connected $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{M}$ then $A\overrightarrow{\mathbf{n}} \geq \overrightarrow{\mathbf{n}}$.*

Proof. Let $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k)$. Suppose that $\Pi : \mathcal{N} \to \mathcal{M}$ defines the given inflation of $\mathcal{M}$. Since $\mathcal{N}$ is an $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{M}$, $\Pi^{-1}(C_i)$ has n_i cells, for all $i \in \mathbf{k}$. For each $i, j \in \mathbf{k}$, there are a_{ij} outputs from C_i to C_j. By the definition of inflation, there are a_{ij} outputs from $\Pi^{-1}(C_i)$ to each cell of $\Pi^{-1}(C_j)$. Hence C_i has a total of $\sum_{j=1, j\neq i}^{k} a_{ij} n_j$ outputs ($a_{ii} = 0$, no self loops). Since $\mathcal{N}$ is strongly connected, each cell of $\Pi^{-1}(C_i)$ must have at least one output. Thus, $n_i \leq \sum_{j=1}^{k} a_{ij} n_j$. This is true for all $i \in \mathbf{k}$. Therefore, $A\overrightarrow{\mathbf{n}} \geq \overrightarrow{\mathbf{n}}$. $\square$

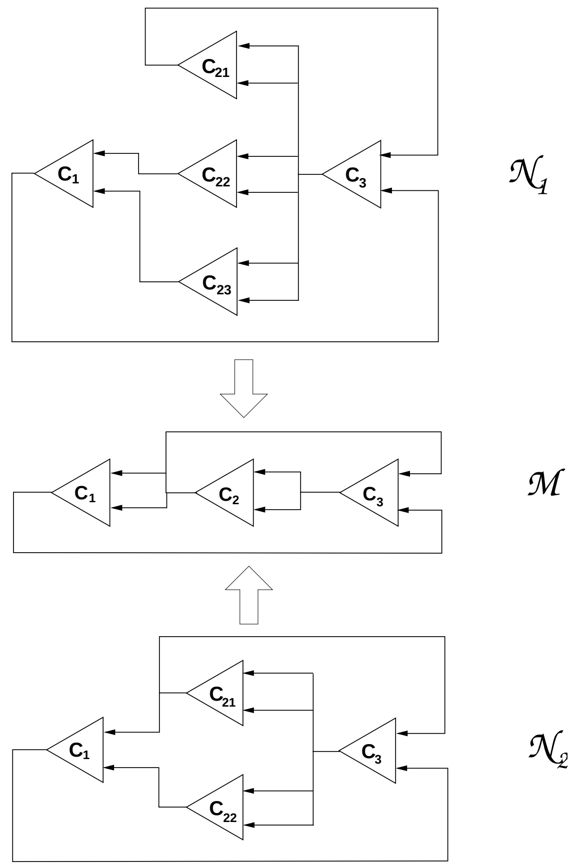

Figure 6.3: $\mathcal{N}_1$ is a strongly connected *3-fold* simple inflation of $\mathcal{M}$; $\mathcal{N}_2$ is a strongly connected *2-fold* simple inflation of $\mathcal{M}$.

Lemma 6.1.5. *Suppose that $\mathcal{M}$ is a strongly connected network containing a cell C with $p > 1$ outputs. Then for $1 < q \leq p$, there is a q-fold strongly connected simple inflation of $\mathcal{M}$ at C.*

Proof. Let the cell set of $\mathcal{M}$ be $\mathcal{C} = \{C_1, \cdots, C_k\}$. By relabelling of cells, assume that $C = C_1$. Let I be the set of cells that have at least one output going to C_1 and $O = \{C_{i_1}, \cdots, C_{i_{\#O}}\}$ be the set of cells that receive inputs from C_1. If a cell receives multiple (say d) inputs from C_1 then we repeat that cell d times in the set O. Thus, $p = \#O = \sum_{j=2}^{k} a_{1j}$. Let $1 < q \leq p$. We construct a *q-fold* simple inflation $\mathcal{N}$ of

$\mathcal{M}$ at C_1 which is strongly connected. Inflate C_1 by q cells, $C_{11}, \cdots, C_{1q}$. Draw connections from $C_{11}, \cdots, C_{1n}$ to cells in O so that all the inputs of cells in O are filled and each C_{1j}, $j \in \mathbf{q}$ has at least one output. For example, $C_{1j} \to C_{i_j}$, $j \in \mathbf{q}$, $C_{1q} \to C_{i_{q+1}}, \cdots, C_{i_{\#O}}$. We show that $\mathcal{N}$ is strongly connected. For $i \in I, j \in \mathbf{q}$, draw a_{i1} connections from C_i to C_{1j}, and for $i, j \neq 1$, draw a_{ij} connections from C_i to C_j (there is a unique choice to draw these connections). It suffices to consider three cases.

(1) if $i, j \in \{2, \cdots, k\}$ and $C_i \rightsquigarrow^{\mathcal{M}}_{p,\mathcal{C}\setminus C_1} C_j$ then $C_i \rightsquigarrow^{\mathcal{N}}_{p} C_j$.

(2) suppose $i, j \in \{2, \cdots, q\}$ and there is a path

$$C_i \to \cdots \to C_1 \to C_{r_1} \to \cdots \to C_1 \to C_{r_t} \to \cdots \to C_j$$

from C_i to C_j such that $r_u \neq r_v$, $u \neq v$. In $\mathcal{N}$, all the inputs of cells in O are filled and $C_{r_u} \in O$. By the construction of $\mathcal{N}$, for each $u \in \mathbf{t}$, there exists $s_u \in \mathbf{q}$ such that there is a connection from C_{1s_u} to C_{r_u}. Thus, $\mathcal{N}$ has a path $C_i \to \cdots \to C_{1s_1} \to C_{r_1} \to \cdots \to C_{1s_t} \to C_{r_t} \to \cdots \to C_j$. This shows that $C_i \rightsquigarrow^{\mathcal{N}}_{p} C_j$.

(3) for each $i, j \in \mathbf{q}$, there are $u_i, u_j \in \{2, \cdots, k\}$ such that $C_{1i} \to C_{u_i}$ and $C_{u_j} \to C_{1j}$. From previous parts, $C_{u_i} \rightsquigarrow^{\mathcal{N}}_{p} C_{u_j}$. Therefore, $C_{1i} \rightsquigarrow^{\mathcal{N}}_{p} C_{1j}$.

Thus, $\mathcal{N}$ is strongly connected. $\square$

In Figure 6.3, we give two examples illustrating the construction of a strongly connected, simple inflation as described in Lemma 6.1.5.

Given $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k)$, $\overrightarrow{\mathbf{m}} = (m_1, \cdots, m_k) \in \mathbb{N}^k$, we write $\overrightarrow{\mathbf{m}} < \overrightarrow{\mathbf{n}}$ if $m_i \leq n_i$, for all $i \in \mathbf{k}$ and $m_j < n_j$, for some $j \in \mathbf{k}$.

Let $\mathcal{M}$ be a strongly connected coupled cell network which is *not* a one-input unidirectional ring, with adjacency matrix A. Let $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k) \in \mathbb{N}^k$ satisfy $A\overrightarrow{\mathbf{n}} \geq \overrightarrow{\mathbf{n}}$. Let $\mathcal{N}$ be a strongly connected $\overrightarrow{\mathfrak{m}}$-inflation of $\mathcal{M}$ defined by $\Pi : \mathcal{N} \to \mathcal{M}$ where $\overrightarrow{\mathbf{m}} = (m_1, \cdots, m_k) \in \mathbb{N}^k$ satisfies $\overrightarrow{\mathbf{m}} < \overrightarrow{\mathbf{n}}$. By Lemma 6.1.4, $A\overrightarrow{\mathbf{m}} \geq \overrightarrow{\mathbf{m}}$. Let $\mathcal{D} = \{\Pi^{-1}(C_i) \mid i \in \mathbf{k}\}$ be the cell set of the network $\mathcal{N}$. Let $I = \{i \in \mathbf{k} \mid m_i < n_i\}$ and $\mathcal{D} = \mathcal{D}_1 \cup (\mathcal{D} \setminus \mathcal{D}_1)$, where $\mathcal{D}_1 = \{\Pi^{-1}(C_i) \mid i \in I\}$.

Lemma 6.1.6. *(Notation and assumptions as above) There exists a cell $C \in \mathcal{D}_1$ such that C has more than one output.*

Proof. We will prove the result by contradiction. Assume that all the cells in $\mathcal{D}_1$ have exactly one output. Note that $\mathcal{D} \setminus \mathcal{D}_1 \neq \emptyset$ otherwise $\mathcal{M}$ is a one-input unidirectional ring.

Claim 1: For each $i \in I$, there exists $j \in I$ such that $a_{ij} > 0$.

Proof of *Claim 1*: Suppose that there exists $i \in I$ such that $a_{ij} = 0$, for all $j \in I$. Then $m_i = \sum_{j \in \mathcal{D} \setminus \mathcal{D}_1} a_{ij} n_j$ (for if, $m_i < \sum_{j \in \mathcal{D} \setminus \mathcal{D}_1} a_{ij} n_j$, then there exists some $C \in \Pi^{-1}(C_i)$ with more than one output). Also, we have $n_i \leq \sum_{j \in \mathcal{D} \setminus \mathcal{D}_1} a_{ij} n_j$. Therefore, $m_i = n_i$, which is a contradiction.

Claim 2: There is no chain contained in $\mathcal{D}_1$.

Proof of *Claim 2*: Let $\widehat{C}$ be a chain in $\mathcal{D}_1$. Since each cell in the chain $\widehat{C}$ has exactly one output, there is no path from E to F, for all cells $E \in \widehat{C}, F \in \mathcal{D} \setminus \mathcal{D}_1$ which contradicts the fact that $\mathcal{N}$ is a strongly connected network.

Claim 3: If $D_1, D_2 \in \Pi^{-1}(C_i)$ for some $i \in I$, then there is no path from D_2 to D_1

which is contained in $\mathcal{D}_1$.

Proof of *Claim 3*: Suppose $D_2 \to C_1^1 \to \cdots \to C_s^1 \to D_1$ is a path from D_2 to D_1 and that $C_i^1 \in \mathcal{D}_1$ for $i \in \mathbf{s}$. Since $D_1, D_2 \in \Pi^{-1}(C_i)$, $i \in I$, there is, by *Claim 1*, a path

$$C_1^2 \to \cdots \to C_s^2 \to D_2 \to C_1^1 \to \cdots \to C_s^1 \to D_1,$$

where $C_i^1, C_i^2 \in \Pi^{-1}(C_{j_i})$ for some $j_i \in I$. Since C_1^1 receives an input from D_2 and $C_1^1, C_1^2 \in \Pi^{-1}(C_{j_1})$, C_1^2 receives an input from D_3 where $D_3 \in \Pi^{-1}(C_i)$. Since there is no chain contained in $\mathcal{D}_1$, by *Claim 2*, D_3 is distinct from D_1, D_2. Iterating this construction, we see that $\Pi^{-1}(C_i)$ is an infinite set, which is a contradiction.

Let

$$\mathcal{A} = \{D \in \mathcal{D}_1 \mid \textit{The output of } D \textit{ is to a cell in } \mathcal{D} \setminus \mathcal{D}_1\}.$$

Since $\mathcal{M}$ is strongly connected, $\mathcal{A} \neq \emptyset$. Let $\mathcal{A} = \{D_1, \cdots, D_s\}$. By *Claim 1*, for each $i \in \mathbf{s}$, there exists $\widetilde{D}_i \in \Pi^{-1}(\Pi(D_i))$ such that there is a connection from $\widetilde{D}_i$ to some cell in $\mathcal{D}_1$. Let $\widetilde{\mathcal{A}} = \{\widetilde{D}_1, \cdots, \widetilde{D}_s\}$. Then $\mathcal{A} \cap \widetilde{\mathcal{A}} = \emptyset$. Since we assumed that the cells in $\mathcal{D}_1$ have exactly one output, $\widetilde{D}_i \notin \mathcal{D} \setminus \mathcal{D}_1$, for each $i \in \mathbf{s}$ and there is a path from $\widetilde{D}_i$ to some cell in $\mathcal{D} \setminus \mathcal{D}_1$, the path must pass through D_j, for some $j \in \mathbf{s}$. Thus, there is a map $\sigma : \mathbf{s} \to \mathbf{s}$ such that $\widetilde{D}_i \rightsquigarrow^{\mathcal{N}}_{p,\mathcal{D}_1} D_{\sigma(i)}$, for $i \in \mathbf{s}$. Also note that $\sigma(i) \neq i$, for all $i \in \mathbf{s}$ else, $\widetilde{D}_i \rightsquigarrow^{\mathcal{N}}_{p,\mathcal{D}_1} D_i$, which contradicts *Claim 3*. Thus, for any $i \in \mathbf{s}$, the set $\{D_{\sigma^r(i)} \mid r = 0, 1, \cdots, s\} \subseteq \mathcal{A}$ consist of distinct elements, otherwise, there is a chain in $\mathcal{D}_1$. Thus, $\#\mathcal{A} > s$, which is a contradiction. Hence, there exists a cell $C \in \mathcal{D}_1$ having more than one output. $\square$

Lemma 6.1.7. *Let* $\mathcal{M}$ *be a strongly connected coupled cell network with at least one cell having more than one output. Let* A *be the adjacency matrix of* $\mathcal{M}$.

Let $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k)$ *be such that* $A\overrightarrow{\mathbf{n}} \geq \overrightarrow{\mathbf{n}}$*. Then there is a sequence of* k*-tuples,* $\overrightarrow{\mathbf{n}}^0 = (1, \cdots, 1) < \overrightarrow{\mathbf{n}}^1 < \overrightarrow{\mathbf{n}}^2 < \cdots < \overrightarrow{\mathbf{n}}^p = \overrightarrow{\mathbf{n}}$ *and simple inflations* $\mathcal{M} = \mathcal{M}_0 \Longleftarrow_S \mathcal{M}_1 \Longleftarrow_S \cdots \Longleftarrow_S \mathcal{M}_p$ *where* $\mathcal{M}_r$ *is a strongly connected* $\overrightarrow{\mathbf{n}}^r$*-inflation of* $\mathcal{M}$*, for* $r \in \mathbf{p}$*.*

Proof. Let $\mathcal{C} = \{C_1, \cdots, C_k\}$ be the cell set of $\mathcal{M}$. Suppose that for some $r \in \mathbb{N}$, we have obtained the sequence $\overrightarrow{\mathbf{n}}^0 < \overrightarrow{\mathbf{n}}^1 < \cdots < \overrightarrow{\mathbf{n}}^{r-1} \leq \overrightarrow{\mathbf{n}}$ and $\mathcal{M}_0 = \mathcal{M} \Longleftarrow_S \mathcal{M}_1 \Longleftarrow_S \cdots \Longleftarrow_S M_{r-1}$. If $\overrightarrow{\mathbf{n}}^{r-1} = \overrightarrow{\mathbf{n}}$, set $p = r-1$ and we are done. Else, assume that $\overrightarrow{\mathbf{n}}^{r-1} < \overrightarrow{\mathbf{n}}$. We will find $\overrightarrow{\mathbf{n}}^r \in \mathbb{N}^k$ such that $\overrightarrow{\mathbf{n}}^{r-1} < \overrightarrow{\mathbf{n}}^r \leq \overrightarrow{\mathbf{n}}$ and a strongly connected simple inflation $\mathcal{M}_r$ of $\mathcal{M}_{r-1}$ such that $\mathcal{M}_r$ is an $\overrightarrow{\mathbf{n}}^r$-inflation of $\mathcal{M}$.

Let $\overrightarrow{\mathbf{n}}^{r-1} = (n_1^{r-1}, \cdots, n_k^{r-1})$. Let $\Pi_{r-1} : \mathcal{M}_{r-1} \to \mathcal{M}$ define the inflation with

$$\Pi_{r-1}^{-1}(C_i) = \{C_{i1}^{r-1}, \cdots, C_{in_i^{r-1}}^{r-1}\},$$

for all $i \in \mathbf{k}$. For $i \in \mathbf{k}$, $j \in \mathbf{n_i^{r-1}}$, let d_i^j denote the number of outputs from the cell C_{ij}^{r-1}. Since $\overrightarrow{\mathbf{n}}^{r-1} < \overrightarrow{\mathbf{n}}$, by Lemma 6.1.6 there exists i, j such that $n_i^{r-1} < n_i$ and $d_i^j > 1$. Let $n = \min\{d_i^j, n_i - n_i^{r-1} + 1\}$. Using Lemma 6.1.5, we may construct a strongly connected *n-fold* simple inflation $\mathcal{M}_r$ of $\mathcal{M}_{r-1}$ at C_{ij}^{r-1}. Set $n_i^r = n + n_i^{r-1} - 1$, $n_u^r = n_u^{r-1}$, for $u \in \mathbf{k} \setminus \{i\}$, $\overrightarrow{\mathbf{n}}^r = (n_1^r, \cdots, n_k^r)$. Thus, noting Remark 6.0.14, $\mathcal{M}_r$ is a strongly connected $\overrightarrow{\mathbf{n}}^r$-inflation of $\mathcal{M}$. □

Lemma 6.1.8. *Let* $\mathcal{M}$ *be a one-input unidirectional ring with* k *cells* $C_1, \cdots, C_k$ *and adjacency matrix* A*. Let* $\mathcal{N}$ *be a strongly connected* $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k)$*-inflation of* $\mathcal{M}$*. Then*

(1) $n_1 = n_2 = \cdots = n_k$*.*

(2) $\mathcal{N}$ is a one-input unidirectional ring with nk cells, where $n = n_1$.

Proof. By relabelling of cells, we can assume that $\mathcal{M} = C_1 \to C_2 \to \cdots \to C_k \to C_1$. Let $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k) \in \mathbb{N}^k$. By Lemma 6.1.4, $A\overrightarrow{\mathbf{n}} \geq \overrightarrow{\mathbf{n}}$ that is, $n_1 \geq n_k \geq \cdots \geq n_2 \geq n_1$. Therefore, $n_1 = n_2 = \cdots = n_k$. Since each cell of $\mathcal{M}$ has exactly one input, each cell of $\mathcal{N}$ has exactly one input (by definition of inflation). For $\mathcal{N}$ to be strongly connected, it must be a one-input unidirectional ring with nk cells. Let $\Pi : \mathcal{N} \to \mathcal{M}$ defines the inflation. For each $i \in \mathbf{k}$, let $\Pi^{-1}(C_i) = \{C_{i1}, \cdots, C_{in}\}$. Then, by relabelling of cells, $\mathcal{N}$ is of the form $C_{11} \to \cdots \to C_{k1} \to \cdots \to C_{1n} \to \cdots \to C_{kn} \to C_{11}$. □

Theorem 6.1.9. *(no self loops) Let $\mathcal{M}$ be a strongly connected k cell network with adjacency matrix A. Let $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k) \in \mathbb{N}^k$. There is a strongly connected $\overrightarrow{\mathbf{n}}$-inflation $\mathcal{N}$ of $\mathcal{M}$ if and only if*

$$A\overrightarrow{\mathbf{n}} \geq \overrightarrow{\mathbf{n}}. \tag{6.1.1}$$

Proof. ($\Rightarrow$): By Lemma 6.1.4, if $\mathcal{N}$ is a strongly connected $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{M}$ then (6.1.1) holds.

($\Leftarrow$): Either $\mathcal{M}$ is a one-input unidirectional ring or $\mathcal{M}$ has at least one cell having more than one output. In the latter case, if $\overrightarrow{\mathbf{n}}$ satisfies the (6.1.1), then the construction of $\mathcal{N}$ follows from Lemma 6.1.7. If $\mathcal{M}$ is a one-input unidirectional ring, Lemma 6.1.8 gives the construction of $\mathcal{N}$. □

Example 6.1.10. Let $\mathcal{M}$ be a network with adjacency matrix $A = [a_{ij}]_{i,j\in\mathbf{k}}$ where $a_{ij} = 1 - \delta_{ij}$ (δ denotes the Kronecker delta function). In this case, every cell has exactly one output to every other cell and there are no self inputs. We call $\mathcal{M}$ a

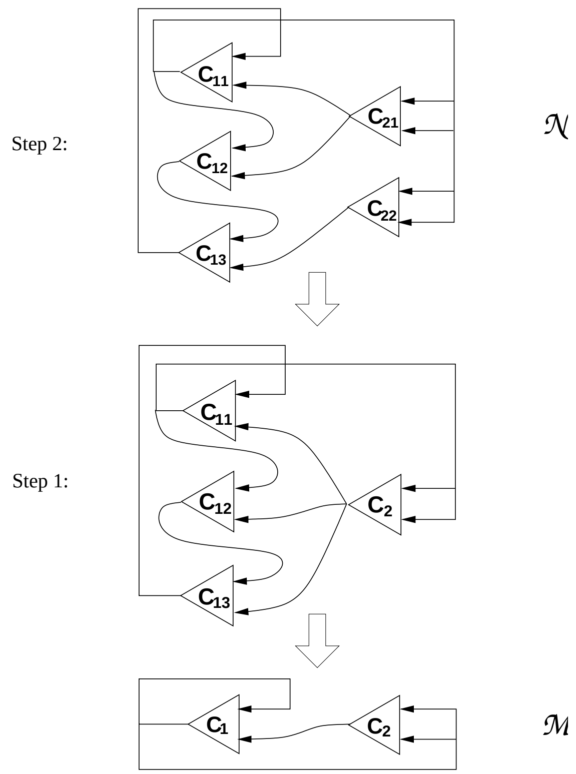

Figure 6.4: The composite of two simple inflations

complete network (in graph theory, the underlying graph is known as a complete graph). For a complete network $\mathcal{M}$, (6.1.1) reduces to

$$\max\{n_i \mid i \in \mathbf{k}\} \leq \sum_{j=1, j\neq i}^{k} n_j.$$

Remark 6.1.11. Suppose $\mathcal{M}$ is a strongly connected network with adjacency matrix A. Let $\overrightarrow{\mathbf{n}} \in \mathbb{N}^k$ satisfy (6.1.1). Theorem 6.1.9 guarantees the existence of a strongly connected $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{M}$. However, an $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{M}$ satisfying (6.1.1) need not be strongly connected. For example, let $\mathcal{M}$, $\mathcal{M}_i$, $i \in \bar{\mathbf{9}}$ be as in Example 6.0.11. The adjacency matrix A of the network $\mathcal{M}$ is $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and $\overrightarrow{\mathbf{n}} = (2,2) \in \mathbb{N}^2$ satisfies condition (6.1.1). The networks $\mathcal{M}_i$, $i \in \mathbf{9}$ are strongly connected $(2,2)$-inflations of $\mathcal{M}$ but $\mathcal{M}_0$ is a $(2,2)$-inflation of $\mathcal{M}$ which is not strongly connected.

6.1.2 Networks with self loops

We stated Theorem 6.1.9 for networks with no self loops. The theorem also holds for networks with some or all of the cells having self loops. Condition (6.1.1) requires that $\sum_{j=1}^{k} a_{ij} n_j \geq n_i$, for all $i \in \mathbf{k}$. If there is an i such that $a_{ii} > 0$ (that is, the i^{th} cell has a self loop) then the condition

$$\sum_{j=1}^{k} a_{ij} n_j \geq n_i$$

is trivially satisfied. Note that if n_j, $j \neq i$, are fixed, then there is no upper bound for n_i. For the case of networks with self loops, we modify the construction in Lemma 6.1.7 in the following manner. Let

$$S = \{i \in \mathbf{k} \mid a_{ii} > 0\}.$$

The set S consists of indices of the cells that have self loops.

Construction of $\mathcal{N}$

(1) for each $i \in S$, inflate the cell C_i by n_i cells, $C_{i1}, \cdots, C_{in_i}$. Then (a) for $i, j \in S$, draw a_{ii} connections from C_{ij} to $C_{i(j+1)}$, for $j \in \mathbf{k-1}$ and, C_{in_i} to C_{i1}, (b) for $i \in S, j \notin S$, draw a_{ij} connections from C_{i1} to C_j, (c) for $i \notin S, j \in S$, draw a_{ij} connections from C_i to C_{jp} for all $p \in \mathbf{n_j}$, and (d) for $i, j \notin S$, draw a_{ij} connections from C_i to C_j. Thus we have obtained a strongly connected inflation of $\mathcal{M}$ with no self loops. It is easy to see that connections drawn in $(c) - (d)$ are unique but there are many choices for the connections drawn in $(a) - (b)$.

(2) now proceed using the construction in Lemma 6.1.7 to obtain the required $\overrightarrow{\mathbf{n}}$-inflation $\mathcal{N}$ of $\mathcal{M}$.

Example 6.1.12. Let $\mathcal{M}$ be as in Figure 6.4, having adjacency matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. The cell C_1 has a self loop. Figure 6.4 shows the construction of a strongly connected $(3, 2)$-inflation $\mathcal{N}$ of $\mathcal{M}$ along the lines described above. Note that $\overrightarrow{\mathbf{n}} = (3, 2)$ satisfies (6.1.1).

6.2 Networks with multiple input types

Let $\mathcal{M}$ be a coupled cell network with k cells labelled $C_1, \cdots, C_k$. Suppose there are $\ell \geq 1$ input types. Then $\mathcal{M}$ has ℓ edge type adjacency matrices $A_1, \cdots, A_\ell$ and the adjacency matrix A of $\mathcal{M}$ is equal to the sum $A_1 + \cdots + A_\ell$.

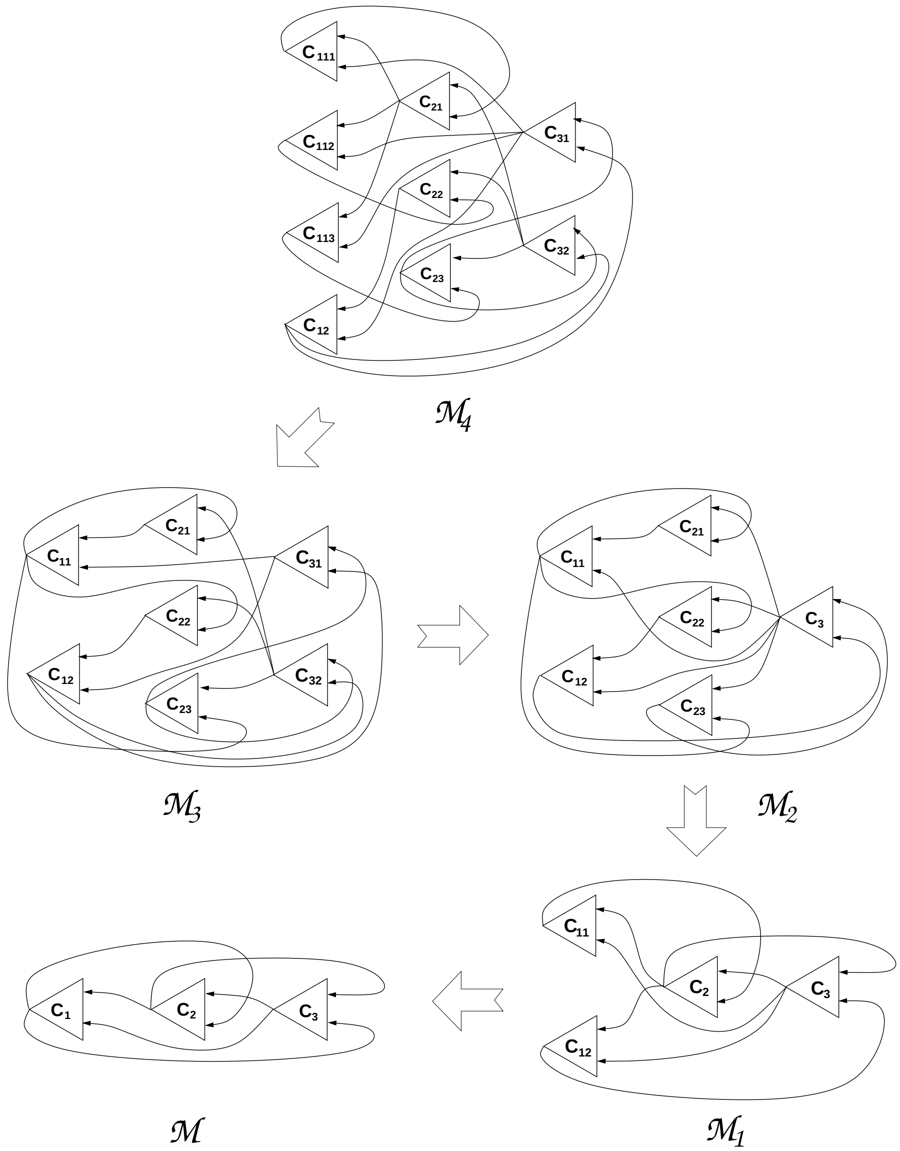

Figure 6.5: A strongly connected $(4, 3, 2)$-inflation $\mathcal{M}_4$, of $\mathcal{M}$ (with symmetric inputs)

Theorem 6.2.1. *Let $\mathcal{M}$ be a strongly connected k cell network with edge type adjacency matrices $A_1, \cdots, A_\ell$. Let $\overrightarrow{\mathbf{n}} = (n_1, \cdots, n_k) \in \mathbb{N}^k$. There is a strongly* *connected* $\overrightarrow{\mathbf{n}}$*-inflation* $\mathcal{N}$ *of* $\mathcal{M}$ *if and only if*

$$A\overrightarrow{\mathbf{n}} = (A_1 + \cdots + A_\ell)\overrightarrow{\mathbf{n}} \geq \overrightarrow{\mathbf{n}}. \tag{6.2.2}$$

Proof. The proof is identical to the proof of Theorem 6.1.9. First assume that all the connection types in $\mathcal{M}$ are identical and construct a strongly connected $\overrightarrow{\mathbf{n}}$-inflation $\mathcal{N}$ of $\mathcal{M}$ defined by $\Pi : \mathcal{N} \to \mathcal{M}$. For each cell in $\Pi^{-1}(C_j)$, there are $\sum_{r=1}^{\ell} a_{ij}^r$ inputs from cells in $\Pi^{-1}(C_i)$. For each $r \in \boldsymbol{\ell}$, regard a_{ij}^r of these connections as connections of type r. $\square$

Remark 6.2.2. The necessary and sufficient condition obtained in Theorem 6.2.1 is independent of the types of edges, it only depends on the total number of inputs and outputs to each cell. Thus, there is no distinction between networks with asymmetric inputs and general networks in the construction of (strongly connected) inflation.

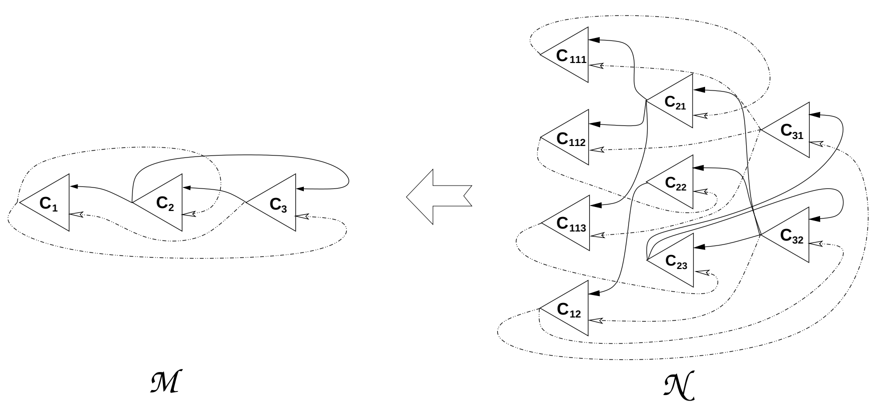

Figure 6.6: A strongly connected $(4, 3, 2)$-inflation $\mathcal{N}$, of $\mathcal{M}$ (asymmetric inputs)

6.3 Examples

Example 6.3.1. (Illustration of Lemma 6.1.7) Let $\mathcal{M}, \mathcal{M}_i$, $i = 1, 2, 3, 4$, be as in Figure 6.5. The network $\mathcal{M}$ is a three-cell bi-directional ring with adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The vector $\overrightarrow{\mathbf{n}} = (4, 3, 2)$ satisfies (6.1.1). In Figure 6.5 we show the construction of a strongly connected $\overrightarrow{\mathbf{n}}$-inflation $\mathcal{N}$ of $\mathcal{M}$, using the method of Lemma 6.1.7.

(1) in $\mathcal{M}$, C_1 has two outputs, let $\overrightarrow{\mathbf{n}}^1 = (2, 1, 1)$ and construct a strongly connected *2-fold* simple inflation $\mathcal{M}_1$ of $\mathcal{M}$, at C_1 by inflating C_1 into two cells C_{11} and C_{12}.

(2) in $\mathcal{M}_1$, C_2 has three outputs, let $\overrightarrow{\mathbf{n}}^2 = (2, 3, 1)$ and construct a strongly connected *3-fold* simple inflation $\mathcal{M}_2$ of $\mathcal{M}_1$, at C_2 by inflating C_2 into three cells C_{21}, C_{22} and C_{23}. At this step, n_2 is achieved.

(3) in $\mathcal{M}_2$, C_3 has five outputs, let $\overrightarrow{\mathbf{n}}^3 = (2, 3, 2)$ and construct a strongly connected *2-fold* simple inflation $\mathcal{M}_3$ of $\mathcal{M}_2$, at C_3 by inflating C_3 into two cells C_{31} and C_{32}. At this step, n_3 is achieved.

(4) in $\mathcal{M}_3$, C_{11} has three outputs, let $\overrightarrow{\mathbf{n}}^4 = (4, 3, 2)$ and construct a strongly connected *3-fold* simple inflation $\mathcal{M}_4$ of $\mathcal{M}_3$, at C_{11} by inflating C_{11} into three cells C_{111}, C_{112}, and C_{113}.

Thus, $\overrightarrow{\mathbf{n}}^4 = \overrightarrow{\mathbf{n}}$ and $\mathcal{N} = \mathcal{M}_4$ is a strongly connected $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{M}$. This serves as a good example to illustrate how the number of synchrony classes may rapidly increase with inflation. The network $\mathcal{M}$ has 2, $\mathcal{M}_1$ has 4, $\mathcal{M}_2$ has 11, $\mathcal{M}_3$ has 22, and $\mathcal{M}_4$ has over 60 non-trivial synchrony classes.

Example 6.3.2. Figure 6.6 shows a strongly connected $(4, 3, 2)$-inflation $\mathcal{N}$ of $\mathcal{M}$ with asymmetric inputs. The only difference between $\mathcal{M}, \mathcal{M}_4$ in Figure 6.5 and $\mathcal{M}$, $\mathcal{N}$ in Figure 6.6 is that we have changed the type of the second input to each cell. If we permute the inputs of any cell in $\mathcal{N}$, we still have an inflation of $\mathcal{M}$ though the new network may no longer be dynamically equivalent to $\mathcal{N}$.

Remark 6.3.3. Theorem 6.2.1 extends immediately to general networks of non-identical cells where the number of input types can depend on the cell class. The statement is identical (with A defined as the adjacency matrix of the network). For the proof, it suffices to note that the construction of the strongly connected inflation $\mathcal{N}$ of the network $\mathcal{M}$ given in Lemma 6.1.7, only requires information about the number of connections and connection type. The class of the cells plays no role in the construction.

Chapter 7

Switching in Heteroclinic Cycles

7.1 Introduction

Let $e_1, \cdots, e_n$ be hyperbolic equilibria of saddle type of a vector field $x' = f(x)$, where $f : \mathbb{R}^m \to \mathbb{R}^m$. If there exist trajectories $\Gamma_1(t), \cdots, \Gamma_n(t)$ such that for $i = 1, \cdots, n$,

$$\lim_{t\to\infty} \Gamma_i(t) = e_{i+1},$$

$$\lim_{t\to\infty} \Gamma_i(-t) = e_i,$$

with the convention that $e_{n+1} = e_1$ then the union of trajectories (Γ_i, e_i) is known as a heteroclinic network.

Krupa and Melbourne [36, 37] have found necessary and sufficient conditions for the asymptotic stability of the heteroclinic network in the presence of symmetry. But their proof depends mainly on the assumption that for each $i = 1, \cdots, n$, there is a flow-invariant subspace P_i such that $W^u(e_i) \subset P_i$ and e_{i+1} is a sink in P_i.

This condition guarantees robustness of the heteroclinic network for vector fields for which P_i are flow-invariant. Another remark is that the necessary and sufficient condition does not depend on the radial directions. Heteroclinic networks occur commonly in dynamical systems with symmetry and trivially are robust under perturbations that preserve the symmetry. They have been found to be relevant in a number of physical phenomena that include population dynamics [29], ecological models [28], in differential equations that are symmetric (or equivariant) [19, 21] and networks of coupled oscillators [10]. Heteroclinic cycles or networks occur in coupled cell systems and population dynamics. Ashwin *et al.* [13] and Broer *et al.* [16, 15] analyze the unstable attractors and heteroclinic cycles between them that can occur in global networks of pulse-coupled oscillators. Kirk and Silber [34] gave an interesting example where there are two heteroclinic networks with a heteroclinic connection in common and discuss about the competition between the two cycles. There are many examples relating to switching dynamics when the vector field is symmetric, given in [5, 30].

Recently Homburg and Knobloch [30] proved an interesting result about switching in a heteroclinic network in $\mathbb{R}^5$ using equivariant dynamics. We are interested in constructing dynamics on a coupled cell network without using equivariance; instead extracting information from the network architecture. We have an example of a 3-cell network with 2-dimensional dynamics on each cell – see [4]. The example illustrates some interesting dynamics that can occur on an asymptotically stable robust heteroclinic network. In addition, there is forward switching behavior between the cycles. There is also numerical evidence of interesting bifurcation behavior in this network.

Consider a vector field on $\mathbb{R}^m$ given by

$$x' \;=\; f(x). \tag{7.1.1}$$

Suppose there is a heteroclinic network Γ consisting of heteroclinic connections $\gamma_1, \cdots, \gamma_n$ joining a set of equillibria. We now give the definition of switching ([30]) in a heteroclinic network. Define the connectivity matrix $C = [c_{ij}]$ where $c_{ij} = 1$ if the end point (the ω-limit set) of γ_i is equal to the starting point (the α-limit set) of γ_j, else $c_{ij} = 0$, for $i = 1, \cdots, n$. Define the following set of symbolic sequences

$$\Sigma_C = \{\kappa = (\kappa_i) \in \{1, \cdots, n\}^{\mathbb{Z}} \mid c_{\kappa_i \kappa_{i+1}} = 1\},$$

$$\Sigma_C^+ = \{\kappa = (\kappa_i) \in \{1, \cdots, n\}^{\mathbb{N}} \mid c_{\kappa_i \kappa_{i+1}} = 1\}.$$

Let N be a tubular region around Γ. Take cross-sections S_i transverse to each γ_i, $i = 1, \cdots, n$ and let $\mathbf{R}$ be the first return map on $S := \cup_{i=1}^n S_i$. Let $\kappa \in \Sigma_C$. We call a *trajectory* T of (7.1.1) a realization of κ in N if there exists a point $x_\kappa \in T$ such that $\mathbf{R}^i(x_\kappa) \in S_{\kappa_i}$, $i \in \mathbb{Z}$. In other words, we say that a realization of a sequence κ is a trajectory that follows the heteroclinic connections in the order prescribed by κ. For $\kappa \in \Sigma_C^+$, we can similarly define a *forward trajectory.*

Definition 7.1.1. A heteroclinic network Γ is a *switching* (forward) if for each $\kappa \in \Sigma_C$ ($\kappa \in \Sigma_C^+$) and each tubular neighbourhood N of Γ, there exists a realization of κ in N.

7.2 Example

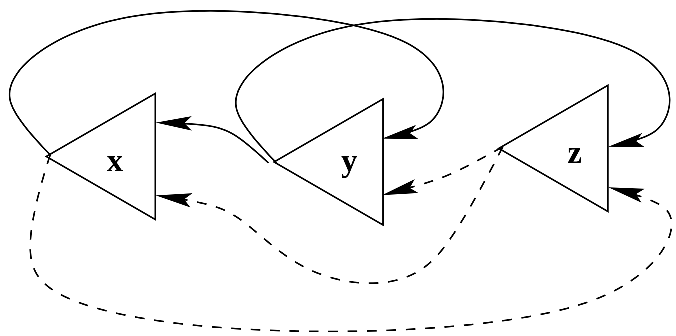

Figure 7.1: The network $\mathcal{M}$

Consider the network architecture $\mathcal{M}$ with 3 cells and 2 asymmetric inputs to each cell (Figure 7.1). Assume that the cells are identical and the phase space of each cell is S. Let $\mathcal{F} \in \mathcal{M}$ be modelled by $f : S \times S \times S \rightarrow TS$ so that the coupled cell system is

$$\mathbf{x}' = f(\mathbf{x}; \mathbf{y}, \mathbf{z}), \ \ \mathbf{y}' = f(\mathbf{y}; \mathbf{x}, \mathbf{z}), \ \ \mathbf{z}' = f(\mathbf{z}; \mathbf{y}, \mathbf{x}).$$

Case 1: $S = \mathbb{R}$.

Let X_f denote the associated vector field on $\mathbb{R}^3$. Let p be an equillibrium of X_f. If $A = \frac{\partial f}{\partial \mathbf{x}}(p)$, $B = \frac{\partial f}{\partial \mathbf{y}}(p)$ and $C = \frac{\partial f}{\partial \mathbf{z}}(p)$. Then the linearized system has the Jacobian given by

$$\begin{pmatrix} A & B & C \\ B & A & C \\ C & B & A \end{pmatrix},$$

which is a 3×3 matrix with eigenvalues given by $A - B, A - C, A + B + C$. By choosing the appropriate A, B, C, we can construct a vector field f such that the system support heteroclinic cycles but there is no possibility of switching in this case since the invariant subspaces are of codimension one. For more details about the construction, we refer to [4, Section 5.3]. If the inputs are symmetric, then the eigenvalues are given by $A-B, A-B, A+2B$. In this case, there is no choice of A, B, C (and therefore f) that can support heteroclinic cycles.

Case 2: $S = \mathbb{R}^2$.

The total phase space is six-dimensional. When the phase space is two-dimensional, A, B, C are 2×2 matrices, and so the eigenvalues of the Jacobian are given by the eigenvalues of $A-B, A-C, A+B+C$. Thus there is a possibility of choosing the vector field so that there are complex eigenvalues. The synchrony subspaces of $\mathcal{M}$ are $\{\mathbf{x} = \mathbf{y} = \mathbf{z}\}$, $\{\mathbf{x} = \mathbf{y}\}$, $\{\mathbf{x} = \mathbf{z}\}$. With two-dimensional cell dynamics, the constructions are simpler since one-dimensional connections generically do not intersect in the four-dimensional synchrony subspaces $\{\mathbf{x} = \mathbf{y}\}$ and $\{\mathbf{x} = \mathbf{z}\}$. Complex eigenvalues may occur for linearizations at equilibria in the maximal synchrony subspace and with associated eigenspaces transverse to the maximal synchrony subspace $\{\mathbf{x} = \mathbf{y} = \mathbf{z}\}$. Since the synchrony subspaces $\{\mathbf{x} = \mathbf{y}\}$ and $\{\mathbf{x} = \mathbf{z}\}$ are of codimension two, there is also the possibility of heteroclinic switching. The example points out interesting dynamics that can occur in a coupled cell network with a simple architecture. Let $f = (f_1, f_2)$ be given by

$$
\begin{aligned}
f_1(\mathbf{x}; \mathbf{y}, \mathbf{z}) \;=\;& x_1(1 - x_1^2 - y_1^2 - z_1^2) + \frac{1}{5}(y_1^3 + z_1^3) + \frac{1}{6} y_1 (y_1 - z_1) \\
&+ \frac{3}{2} x_1 (y_2 - z_2) - \frac{1}{4}(2x_2 - y_2 - z_2)(y_1^2 + z_1^2) + y_1^2 - z_1^2
\end{aligned}
$$

$$\begin{aligned} & +(y_2^3 + z_2^3), \\ f_2(\mathbf{x};\mathbf{y},\mathbf{z}) &= -x_2 + \frac{1}{10}(y_2 + z_2) - 3x_1(y_1 - z_1) + \frac{5}{6}x_1(y_2 - z_2) \\ & +y_1^2 - z_1^2. \end{aligned}$$

There are three hyperbolic equillibria $\pm\mathbf{p}$, $\mathbf{O}$ on the two-dimensional synchrony subspace $\{\mathbf{x} = \mathbf{y} = \mathbf{z}\}$ given by

$$\mathbf{p} = \left(\sqrt{\frac{1}{2.6}}, 0, \sqrt{\frac{1}{2.6}}, 0, \sqrt{\frac{1}{2.6}}, 0\right), \ \mathbf{O} = (0, 0, 0, 0, 0, 0).$$

The equillibria $\pm\mathbf{p}$ have one-dimensional unstable manifolds and complex contracting eigenvalues contained in the four-dimensional synchrony subspaces $\{\mathbf{x} = \mathbf{y}\}$ and $\{\mathbf{x} = \mathbf{z}\}$ and transverse to $\{\mathbf{x} = \mathbf{y} = \mathbf{z}\}$. There are heteroclinic connections γ_1, γ_2 from $\mathbf{p}$ to $-\mathbf{p}$, and connections γ_3, γ_4 from $-\mathbf{p}$ to $\mathbf{p}$. The eigenvalues at each equillibria $\pm\mathbf{p}$ satisfy the necessary and sufficient condition of Krupa and Melbourne [36, 37], and thus the heteroclinic network constituting the cycles γ_i, $i = 1, \cdots, 4$ is asymptotically stable and robust under perturbations that preserve the network architecture $\mathcal{M}$. The heteroclinc network and the dynamical behavior – switching mechanism and asymptotic stability is shown in Figures 7.2, 7.3.

7.3 Switching in a heteroclinic network

Consider a coupled cell network with architecture $\mathcal{M}$ shown in Figure 7.1 (assume identical cells). Let $(\mathbf{x}', \mathbf{y}', \mathbf{z}') = F(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be the associated dynamical system. Assume that the system has two hyperbolic equillibria $P, Q \in \{\mathbf{x} = \mathbf{y} = \mathbf{z}\}$. We

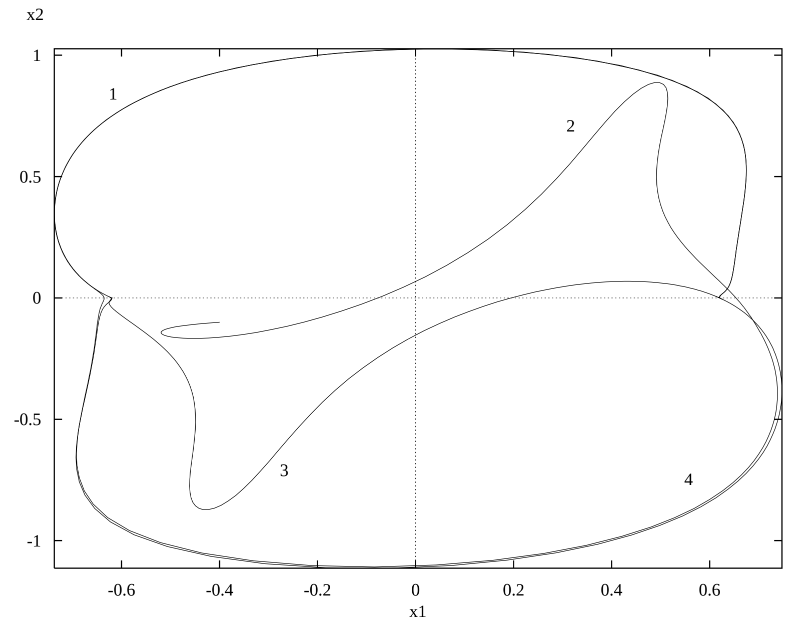

Figure 7.2: Plot shows the time evolution of $\mathbf{x}$ and the trajectories close to the cycles γ_i, $i = 1, \cdots, 4$ projected on the (x_1, x_2)-plane.

make the following assumptions on the spectrum of $DF(P)$:

1. two real eigenvalues $-r_{11} < 0$, $-r_{12} < 0$ with eigenspaces contained in $\{\mathbf{x} = \mathbf{y} = \mathbf{z}\}$.

2. two real eigenvalues $\mu_1 > 0$, $-d_1 < 0$ with eigenspaces contained in $\{\mathbf{x} = \mathbf{y}\}$.

3. a pair of complex conjugate eigenvalues $-\lambda_1 \pm i$, $\lambda_1 > 0$ with eigenspace contained in $\{\mathbf{x} = \mathbf{z}\}$. Let $p_1 = \mu_1/\lambda_1$.

Similarly, we make the following assumptions on the spectrum of $DF(Q)$:

1. two real eigenvalues $-r_{21} < 0$, $-r_{22} < 0$ with eigenspaces contained in $\{\mathbf{x} = \mathbf{y} = \mathbf{z}\}$.

2. two real eigenvalues $\mu_2 > 0$, $-d_2 < 0$ with eigenspaces contained in $\{\mathbf{x} = \mathbf{z}\}$.

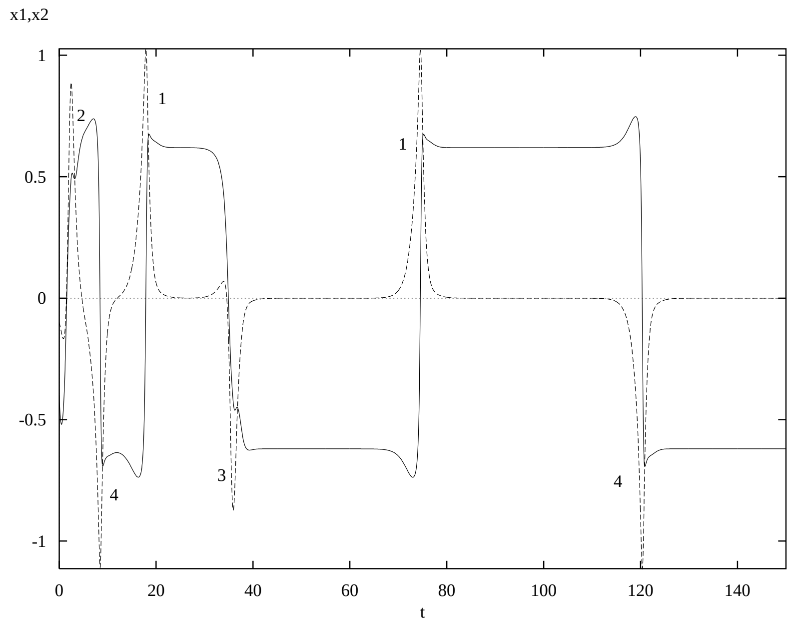

Figure 7.3: The time series of $\mathbf{x}$ coordinates. Switching can be observed between connections and increase in the time to intersect t axes shows the asymptotic stability of the network.

3. a pair of complex conjugate eigenvalues $-\lambda_2 \pm i$, $\lambda_2 > 0$ with eigenspace contained in $\{\mathbf{x} = \mathbf{y}\}$. Let $p_2 = \mu_2/\lambda_2$.

We make the following conditions on the eigenvalues of P and Q.

(**A1**) For $i = 1, 2$; $0 < \mu_i, \lambda_i < r_{i1}, r_{i2}, d_i$. That is, $\mu_i, -\lambda_i \pm i$ are the leading eigenvalues.

We assume that there are heteroclinic trajectories $\Gamma_1^{\pm}$ from P to Q contained in the invariant subspace $\{\mathbf{x} = \mathbf{z}\}$ and heteroclinic trajectories $\Gamma_2^{\pm}$ from Q to P contained in the invariant subspace $\{\mathbf{x} = \mathbf{y}\}$. Since the connections are contained the invariant subspaces, the intersection of the unstable manifold of P and the stable manifold Q is transverse. Similarly, the intersection of the unstable manifold of Q and the stable manifold P is transverse. Also, the trichotomy condition [43, Chapter 6] holds by (**A1**). Thus, using the center manifold reduction for heteroclinic cycles [43, Theorem 6.6], we can reduce the above system to a 3 dimensional system having two hyperbolic equillibria P, Q. The linearized system at P, Q is as follows:

$$\begin{aligned} r' &= -\lambda_1 r & \qquad r' &= -\lambda_2 r, \\ \theta' &= 1 & \qquad \theta' &= 1, \\ z' &= \mu_1 z & \qquad z' &= \mu_2 z, \end{aligned}$$

where $(x, y) \sim (r, \theta)$ using a polar change of coordinates. We construct local 3 dimensional neighbourhoods of P, Q as follows:

$$\begin{aligned} N_P &= \{(r, \theta, z) \mid r < r_P, |z| < \delta, \theta \in [0, 2\pi)\}, \\ N_Q &= \{(r, \theta, z) \mid r < r_Q, |z| < \delta, \theta \in [0, 2\pi)\}. \end{aligned}$$

We define $N_P^+ := N_P \cap \{z > 0\}$, $N_P^- := N_P \cap \{z < 0\}$. Similarly, $N_Q^{\pm}$. We have $N_P \cap \{z = 0\} \subset W^s(P)$, $N_Q \cap \{z = 0\} \subset W^s(Q)$. Suppose that the connection Γ_1^+ from P to Q passes through $(0, 0, \delta) \in \partial N_P$ and $(r_Q, 0, 0) \in N_Q$ and the connection Γ_2^- from Q to P passes through $(0, 0, \delta) \in \partial N_Q$ and $(r_P, 0, 0) \in N_P$. Define the cross-sections H_P^{in}, H_Q^{in}, H_P^{out}, H_Q^{out} as follows:

$$H_P^{in} = \{(r_P, \theta, z) \mid |\theta| < \delta, |z| < \delta\}\ ,\ H_P^{out} = \{(r, \theta, \pm\delta) \mid r < r_P, \theta \in [0, 2\pi)\},$$

$$H_Q^{in} = \{(r_Q, \theta, z) \mid |\theta| < \delta, |z| < \delta\}\ ,\ H_Q^{out} = \{(r, \theta, \pm\delta) \mid r < r_Q, \theta \in [0, 2\pi)\}.$$

Define the local maps $\phi_P : H_P^{in} \setminus W^s(P) \to H_P^{out}, \phi_Q : H_Q^{in} \setminus W^s(Q) \to H_Q^{out}$ and the global connecting diffeomorphisms $\psi_{PQ} : H_P^{out} \to H_Q^{in}, \psi_{QP} : H_Q^{out} \to H_P^{in}$. Using the linearizations at P, Q, we can compute that

$$\begin{aligned}
\phi_P(r_P, \theta, z) &= \left(r_P \left(\frac{z}{\delta}\right)^{\frac{\lambda_1}{\mu_1}}, \theta - \frac{1}{\mu_1} \log\left(\frac{z}{\delta}\right), \delta \right), \\
\phi_Q(r_Q, \theta, z) &= \left(r_Q \left(\frac{z}{\delta}\right)^{\frac{\lambda_2}{\mu_2}}, \theta - \frac{1}{\mu_2} \log\left(\frac{z}{\delta}\right), \delta \right).
\end{aligned}$$

We assume the the connecting maps are affine invertible maps given by

$$\begin{aligned}
\psi_{PQ}(r, \theta, \delta) &= (r_Q, ar\cos(\theta) + br\sin(\theta) + \theta_0, cr\cos(\theta) + d\sin(\theta)), \\
\psi_{QP}(r, \theta, \delta) &= (r_P, a'r\cos(\theta) + b'r\sin(\theta) + \theta_0', c'r\cos(\theta) + d'\sin(\theta)),
\end{aligned}$$

where $(r_Q, \theta_0, 0)$ is the point of intersection of Γ_1 with $W^s(Q)$, and $(r_P, \theta_0', 0)$ is the point of intersection of Γ_2 with $W^s(P)$. For simplicity, we assume $a, d, a', d' = 1$, $b, c, b', c' = 0$, $\theta_0, \theta_0' = 0$. See Remark 7.5.2 for comments about the general case. Let $g_1 = \psi_{PQ} \circ \phi_P : H_P^{in} \setminus W^s(P) \to H_Q^{in}$ and $g_2 = \psi_{QP} \circ \phi_Q : H_Q^{in} \setminus W^s(Q) \to H_P^{in}$. The map $g_1(r_P, \theta, z)$ intersects $W^s(Q)$ if $\theta - \frac{1}{\mu_1} \log\left(\frac{z}{\delta}\right) = n\pi$, $n \in \mathbb{Z}$. For $n \in \mathbb{N}$, define the strips

$$H_n = \{(r_P, \theta, z) \mid |\theta| < \delta,\ n\pi < \theta - \frac{1}{\mu_1} \log\left(\frac{z}{\delta}\right) < (n+1)\pi\}.$$

If n is even, $g_1(H_n) \subset \{z > 0\}$, and if n is odd, $g_1(H_n) \subset \{z < 0\}$. Also, $g_1(\partial H_n) \subset \{z = 0\}$.

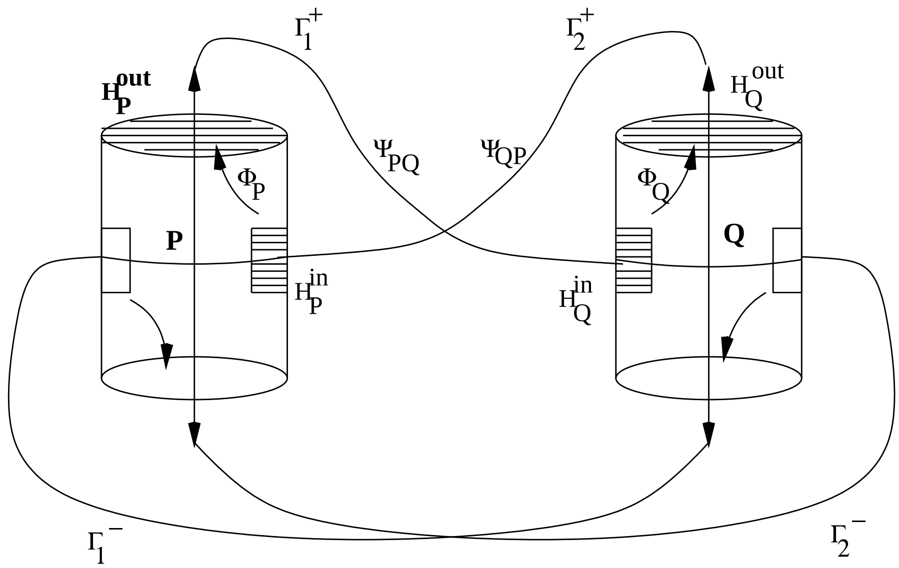

Figure 7.4: Local neighbourhoods and cross-sections at P and Q.

The curves $(r_P, \theta, \delta e^{\mu_1(\theta - n\pi)}) \cap H_P^{in}$ accumulate onto the stable manifold of P, $\{z = 0\}$. Therefore, there exists $N \in \mathbb{N}$ such that $H_n \cap H_P^{in} \neq \emptyset$, for all $n \geq N$. Hence,

$$H_P^{in} = \bigcup_{n=N}^{\infty} (H_n \cap H_P^{in}).$$

Similarly, for $n \in \mathbb{N}$, define the strips

$$U_n = \{(r_Q, \theta, z) \mid |\theta| < \delta,\ n\pi < \theta - \frac{1}{\mu_2} \log\left(\frac{z}{\delta}\right) < (n+1)\pi\}.$$

If n is even, $g_2(U_n) \subset \{z > 0\}$, and if n is odd, $g_2(U_n) \subset \{z < 0\}$. Also, $g_2(\partial U_n) \subset \{z = 0\}$.

The curves $(r_Q, \theta, \delta e^{\mu_2(\theta - n\pi)}) \cap H_Q^{in}$ accumulate onto the stable manifold of Q, $\{z = 0\}$. Therefore, there exists $M \in \mathbb{N}$ such that $U_n \cap H_Q^{in} \neq \emptyset$, for all $n \geq M$.

Hence,

$$H_Q^{in} = \bigcup_{n=M}^{\infty} (U_n \cap H_Q^{in}).$$

7.4 Forward switching

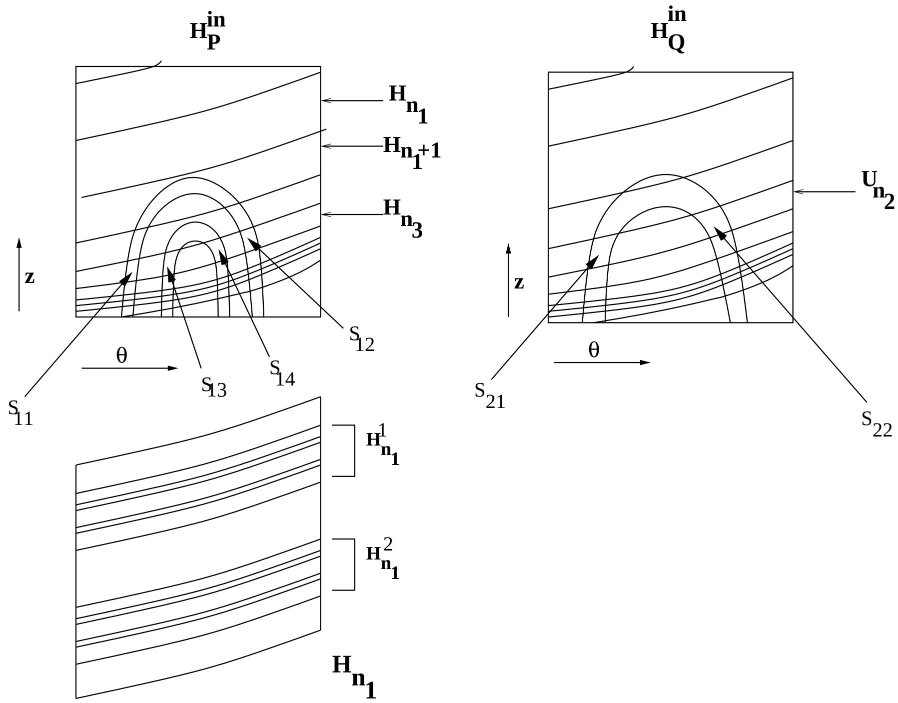

Figure 7.5: Geometrical explanation of forward switching.

Choose the smallest $n_1 \geq N$ such that $e^{\mu_1(\delta - n_1\pi)} < 1$, consider the strip H_{n_1}. Without loss of generality, assume that n_1 is even, then $g_1(H_{n_1}) \subset \{z > 0\}$ and $g_1(H_{n_1+1}) \subset \{z > 0\}$ and $g_1(\partial H_{n_1} \cap \partial H_{n_1+1}) \subset \{z = 0\}$. Note that however small $\delta > 0$ we choose, such n_1 always exist since H_{n_1} accumulate on $W^s(P)$, $\{z = 0\}$.

The g_1 image of H_{n_1} is the region delimited by two half spirals. Since the strips U_n accumulate on $W^s(Q)$, $\{z = 0\}$, there exists $n_2 \geq M$, n_2 even such that

$g_1(H_{n_1})$ intersects U_{n_2} in two disconnected squares S_{21}, S_{22}. Thus we have two disjoint strips $H^1_{n_1}$ and $H^2_{n_1}$ contained in H_{n_1} such that $g_1(H^i_{n_1}) = S_{2i}$, $i = 1, 2$. Since two sides of the squares S_{21}, S_{22} are contained in ∂U_{n_2}, we have $g_2(S_{21})$ and $g_2(S_{22})$ are two disjoint regions delimited by half spirals. Again, since the strips H_n accumulate on $W^s(P)$, $\{z = 0\}$, there exists $n_3 \geq N$, n_3 even such that $g_1(H_{n_1}) \cap U_{n_2}$ intersects H_{n_3} in four disconnected squares S_{1i}, $i = 1, \cdots, 4$. Thus for $i = 1, 2$, we have two disjoint strips $H^{i1}_{n_1}$ and $H^{i2}_{n_1}$ contained in $H^i_{n_1}$ such that $g_2 \circ g_1(H^{11}_{n_1}) = S_{11}$, $g_2 \circ g_1(H^{12}_{n_1}) = S_{12}$, $g_2 \circ g_1(H^{21}_{n_1}) = S_{13}$, $g_2 \circ g_1(H^{22}_{n_1}) = S_{14}$. It is easy to see that H_{n_1+1} is also partitioned into thinner strips.

There is a bijection between the symbols $\{E, O\}$ and $\{P, N\}$, where E is mapped to P and O is mapped to N, that is, even number is identified with positive connection and odd number is identified with negative connection. Thus we have a natural bijection between the one-sided shift on two symbols E, and O, and the path of connections.

Given a path (p_n), we choose the sequence of $H_{n_1}, U_{n_2}, H_{n_3}, U_{n_4, \cdots}$ such that n_i is even if $p_i = P$, and odd if $p_i = N$. Hence we have proved forward switching in the network.

Theorem 7.4.1. *If (**A1**) holds and $p_1 p_2 > 1$, then $\Gamma = \Gamma^{\pm}_1 \cup \Gamma^{\pm}_2$ is a robust asymptotically stable heteroclinic network which is forward switching.*

Proof. The asymptotic stability is clear by analyzing the return maps $g_1 \circ g_2$ and $g_2 \circ g_1$ to the cross-sections H^{in}_P and H^{in}_Q. □

Remark 7.4.2. The condition $p_1 p_2 > 1$ is the necessary and sufficient condition of Krupa and Melbourne [36, Theorem 2.7] for asymptotic stability of heteroclinic network.

7.5 Horseshoes

Lets make the following assumption on the eigenvalues : (**A2**) $p_1p_2 < 1$.

Let

$$U_m^c = \{(r_Q, \theta, z) \mid |\theta| < \delta,\ \theta - \frac{1}{\mu_2}\log(z) = m\pi + \frac{\pi}{2}\}.$$

Lemma 7.5.1. *Consider H_n for fixed n 'even' sufficiently large. Let m be the minimum 'even' such that U_m^c intersects $g_1(H_n)$ at at least two points. Let $S = U_m \cap g_1(H_n)$. The innermost boundary of $g_2(S)$ intersects the upper horizontal boundary of H_i in at least two points, for $i \geq n/\alpha$, where $1 \leq \alpha < 1/p_1p_2$. Moreover, the preimage of the vertical boundaries of $g(H_n) \cap H_i$ are contained in the vertical boundary of H_n.*

Proof. The z coordinate of the upper horizontal boundary of H_i is $\overline{z} = \delta e^{\mu_1(\delta - i\pi)}$. The point $R = (r_P, \theta, z)$ on the innermost boundary of $g_2(S)$ closest to $(r_P, 0, \delta)$ is when $\phi - \frac{1}{\mu_2}\log(Z) = m\pi + \pi/2$. That is, $\hat{r} = \sqrt{\theta^2 + z^2} = z^{p_1p_2}$, where $\psi = n\pi + u$, $z = \delta e^{\mu_1(-\delta-\psi)}$, for some $\pi/2 \leq u \leq \pi$. We want $\overline{z} < \hat{r}$,

$$\begin{aligned} \overline{z} &< \hat{r} \\ e^{\mu_1(\delta - i\pi)} &< e^{\mu_1 p_1 p_2(-\delta - n\pi - u)} \\ K\ e^{\pi\mu_1(i - p_1p_2n)} &> 1. \end{aligned}$$

The term K is a constant, therefore $K\ e^{\pi\mu_1(i-p_1p_2n)} > 1$ means we want $i - p_1p_2n$ to be sufficiently large (positive). Choose $1 \leq \alpha < 1/p_1p_2$ and $i \geq k/\alpha$ then $i - p_1p_2n$ is positive, and for k sufficiently large, $K\ e^{\pi\mu_1(i-p_1p_2n)} > 1$.

The map $g = g_2 \circ g_1$ maps vertical boundaries of H_k to the spirals. The vertical boundaries of $g(H_n) \cap H_i$ are contained in the spirals, hence the preimage is contained in the vertical boundary of H_n. $\square$

Remark 7.5.2. If we take the connecting maps to be general affine invertible maps then the point $\tilde{R} = (r_P, \tilde{\theta}, \tilde{z})$ closest to (r_P, θ_0, δ) has $\tilde{r} = C\hat{r}$, for some constant $C > 0$, $\tilde{r} = \sqrt{\tilde{\theta}^2 + \tilde{z}^2}$. Thus solving for $\overline{z} < \tilde{r}$ gives us the same result as in Lemma 7.5.

Theorem 7.5.3. *For n sufficiently large, H_n contains an invariant Cantor set, Λ_n, on which the map g is topologically conjugate to a full shift on two symbols.*

Proof. The proof is similar to the proof of [48, Theorem 4.8.4]. $\square$

Chapter 8

Conclusions

We conclude the dissertation by pointing out some interesting observations from our work and possible future directions. The notion of dynamical equivalence – *input and output* developed in the dissertation provides a tool for understanding the dynamics of the coupled cell dynamical systems using the network architecture. The results presented here lead to the construction of universal networks, described in Section 5.4. The important observation here is that the number of inputs can be significantly reduced using dynamical equivalence results. This has a straightforward application to electrical engineering where the cost and labour to construct a device having some dynamical behavior can be reduced using these results.

The proofs of Theorem 4.2.6, 5.1.13 are algorithmic in nature and give a systematic way to convert a network into another dynamically equivalent network by a sequence of *simple moves* – changing one input at a time. Similarly, the proof of Theorem 6.2.1 provide a clear algorithmic procedure to construct a strongly connected inflation using a sequence of *simple inflations* – inflating one cell at a time.

Another interesting observation from Theorem 4.2.6 is that for every network $\mathcal{M}$ we have $\mathcal{M} \sim_O \mathcal{M}$. We call this *self equivalence* and illustrate it with the following example.

Example 8.0.4. There may be many ways of achieving self output equivalence. For example, consider the two-cell network $\mathcal{M}$ with asymmetric inputs shown in Figure 8.1.

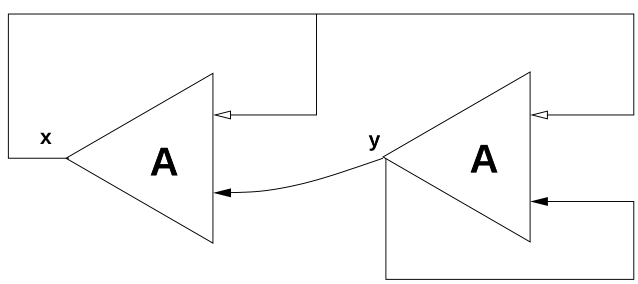

Figure 8.1: A two-cell network $\mathcal{M}$ with asymmetric inputs

Suppose $\mathcal{F} \in \mathcal{M}$ has model f. It can be shown that the two-parameter family defined for $c, d \in \mathbb{R}$ by

$$\begin{aligned} f_{c,d}(x_0; x_1, x_2) = {} & cf(x_0; x_0, x_0) + df(x_0; x_0, x_1) \\ & - (c+d)f(x_0; x_0, x_2) - (c+d)f(x_0; x_1, x_0) \\ & + (1+c+d)f(x_0; x_1, x_2) + df(x_0; x_2, x_0) \\ & - df(x_0; x_2, x_1), \end{aligned}$$

gives all output equivalences $\mathcal{M} \sim_O \mathcal{M}$. For example, if we take $c = 0$, $d = -1/2$, then

$$\begin{aligned} g(x_0;x_1,x_2) &= f_{0,-1/2}(x_0;x_1,x_2) \\ &= \frac{1}{2}(-f(x_0;x_0,x_1)+f(x_0;x_0,x_2)+f(x_0;x_1,x_0) \\ &\quad +f(x_0;x_1,x_2)-f(x_0;x_2,x_0)+f(x_0;x_2,x_1)). \end{aligned}$$

In terms of ordinary differential equations, if the model for a cell is $f(x_0;x_1,x_2) = x_0x_1x_2^2+x_0$, and we define

$$\begin{aligned} g(x_0;x_1,x_2) &= f_{0,-1/2}(x_0;x_1,x_2) \\ &= \frac{1}{2}(-x_0^2x_1^2+x_0^2x_2^2+x_0^3x_1+x_0x_1x_2^2-x_0^3x_2+x_0x_2x_1^2+2x_0), \end{aligned}$$

then $x' = f(x;x,y)$, $y' = f(y;x,y)$ and $x' = g(x;x,y), y' = g(x;x,y)$ have identical dynamics, even though the models f and g are quite different. Note, however, that if f is a linear vector field, or is of the form $f(x;y,z) = au(x;y) + bv(x;z)$, then $f = g$. In particular, it seems we cannot usefully develop this idea using the concept of linear self-equivalence [18].

Continuing with our choice of $c = 0$, $d = -1/2$, define the new cell class $\mathbf{A}^\star$ as in Figure 8.2. Although the new cell is different from the original cell $\mathbf{A}$, when it is incorporated in the network $\mathcal{M}$, it will give the same dynamics.

This construction leads naturally to a number of observations and questions and we conclude by briefly discussing some of these issues.

- To what extent can this process be reversed? That is, given a network of 'complex' cells, when is it equivalent to the same network but built of simpler

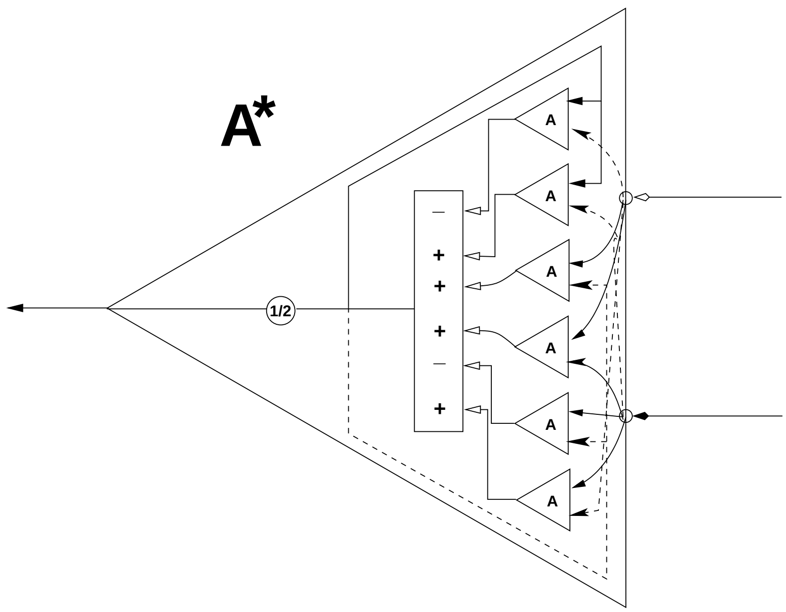

Figure 8.2: The cell $\mathbf{A}^\star$

cells?

- Is there a way of choosing the specific output equivalence so as to protect against failure of individual units comprising the new cells? For example, if we build the network $\mathcal{M}$ from the cells $\mathbf{A}^\star$, what is the effect on network dynamics of the failure of a single $\mathbf{A}$-cell in $\mathbf{A}^\star$?

- Is there an optimal way of choosing the output equivalence so as to minimize the effect of failure of individual units?

- Are there potential applications to numerical analysis (for example, in the solution of partial differential equations)?

- There are also questions related to the effects of *inflation* (see Chapter 6, [4]) on $\mathbf{A}$-cells in $\mathbf{A}^\star$.

- Another potentially interesting question is to extend the notion of input equivalence to allow for nonlinear combinations of inputs. This would seem to be of particular interest for scalar signalling networks and self-loops.

- Determining an estimate for the number of synchrony classes which increases with inflation is also useful. Example 6.0.11 shows that even for a network with a simple architecture, the number of synchrony classes can rapidly increase.

- Another interesting problem is related to inflation of dynamically equivalent networks. More precisely, whether there is a correspondence between the set of inflations of two dynamically equivalent networks? The answer is *NO* and we illustrate it with Example 8.0.5.

Example 8.0.5. Let $\mathcal{M}$ and $\mathcal{N}$ be as in Figure 4.1. The adjacency matrices of $\mathcal{M}$ are $M_0 = I$, $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and $M_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $\mathcal{N}$ are $N_0 = I$, $N_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and $N_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Here, $N_2 = M_1 + M_2 - M_0$. Since both the cells in $\mathcal{M}$ have self loops, $\mathcal{M}$ has an $\overrightarrow{\mathbf{n}} = (n_1, n_2)$-inflation for each $\overrightarrow{\mathbf{n}} \in \mathbb{N}^2$. But the condition for existence of a strongly connected $\overrightarrow{\mathbf{n}}$-inflation of $\mathcal{N}$ is that $n_1 \geq n_2$.

Appendix

8.1 Relation between g and f in Example 5.1.24

Using Examples 5.1.17, 5.1.20, we get

$$\begin{aligned} g(x_0;\overline{x_1,\cdots,x_4},\overline{x_5,x_6}) &= h(x_0;\overline{x_1,\cdots,x_4,x_5,x_6},\overline{x_0,x_0}) \\ &= \sum_{1\le j_1<\cdots<j_4\le 6} e(x_0;\overline{x_{j_1},\cdots,x_{j_4}}) \\ &\quad -2\sum_{1\le j_1<\cdots<j_3\le 6} e(x_0;\overline{x_0,x_{j_1},x_{j_2},x_{j_3}}) \\ &\quad +3\sum_{1\le j_1<j_2\le 6} e(x_0;\overline{x_0,x_0,x_{j_1},x_{j_2}}) \\ &\quad -4\sum_{1\le i_1\le 6} e(x_0;\overline{x_0,x_0,x_0,x_{j_1}}) \\ &\quad +5\ e(x_0;\overline{x_0,\cdots,x_0}). \end{aligned}$$

Using Equation 5.1.7 obtained in Example 5.1.22 (writing e in terms of f), we get the following terms $f(x_0;\overline{x_{j_1},x_{j_2}})$, $f(x_0;\overline{x_0,x_{j_2}})$, $f(x_0;\overline{x_{j_1},x_{j_1}})$, and $f(x_0;\overline{x_0,x_0})$, for $1\le j_1<j_2\le 6$. Table 8.1 gives the coefficients of these terms.

Term	Coefficient
$f(x_0;\overline{x_{j_1},x_{j_2}})$	$\frac{1}{4}(\binom{4}{2}-2\binom{4}{1}+3)=\frac{1}{4}$
$f(x_0;\overline{x_{j_1},x_{j_1}})$	$-\frac{1}{8}(\binom{5}{3}-2\binom{5}{2}+3\binom{5}{1}-4)=-\frac{1}{8}$
$f(x_0;\overline{x_0,x_{j_1}})$	$\frac{1}{4}(-2\binom{5}{2}+3.2.\binom{5}{1}-4.3)=-\frac{1}{2}$
$f(x_0;\overline{x_0,x_0})$	$\frac{1}{4}(3\binom{6}{2}-4.3.\binom{6}{1}+5.6)-\frac{1}{8}(-2\binom{6}{2}+3.2.\binom{6}{2}-4.3.\binom{6}{1}+5.4)=1$

Table 8.1: Coefficients

Thus we obtain the relation (mentioned in Example 5.1.24)

$$
\begin{aligned}
g(x_0;\overline{x_1,\cdots,x_4},\overline{x_5,x_6}) &= \frac{1}{4}\sum_{1\le j_1<j_2\le 6} f(x_0;\overline{x_{j_1},x_{j_2}})+f(x_0;\overline{x_0,x_0}) \\
&\quad -\frac{1}{8}\sum_{1\le j_1\le 6} f(x_0;\overline{x_{j_1},x_{j_1}})-\frac{1}{2}\sum_{1\le j_1\le 6} f(x_0;\overline{x_0,x_{j_1}}).
\end{aligned}
$$

With some more calculations, it can be checked that

$$
\begin{aligned}
x_1' &= g(x_1;\overline{x_1,x_1,x_1,x_2},\overline{x_1,x_2}) = f(x_1;\overline{x_1,x_2}), \\
x_2' &= g(x_2;\overline{x_2,x_2,x_3,x_3},\overline{x_2,x_2}) = f(x_2;\overline{x_2,x_3}), \\
x_3' &= g(x_3;\overline{x_1,x_1,x_2,x_2},\overline{x_3,x_3}) = f(x_3;\overline{x_1,x_2}).
\end{aligned}
$$

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