NON-INTEGRATED DEFECT RELATION FOR MEROMORPHIC MAPS OF COMPLETE KÄHLER MANIFOLDS ENCOUNTERING DIVISORS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Suraizou Sogomé

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ABSTRACT

In this dissertation, we are interested in meromorphic maps of a complete Kähler manifold whose universal covering is biholomorphic to the ball in $\mathbb{C}^m$ into complex projective manifolds. We first give a non-integrated defect for a meromorphic map of the above Kähler manifold into $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces in general position and an application of this non-intetgrated defect to the Gauss map of a complete regular submanifold of $\mathbb{C}^m$.

We then focus on the uniqueness problem, i.e, to find how many hypersurfaces are sufficient to uniquely determine a map which intersects them. The first result provides a complement to the recent result of Min Ru on the defect relation for meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces in general position and the second provides a complement to the result of H. Fujimoto on the unicity theorem for meromorphic maps of a complete Kähler manifold into $\mathbb{P}^n(\mathbb{C})$.

Contents

Chapter 1

Introduction and background

1.1 Preface

Nevanlinna theory is a branch of complex analysis which studies the image of meromorphic maps between complex manifolds. Foundational results are found in classical complex analysis. For example the well-known fundamental theorem of algebra states that: *for every complex polynomial $P(z)$ of degree d and every complex number a, the equation $P(z) = a$ has d solutions on the complex plane, counting multiplicities.* The little Picard Theorem, viewed as a generalization of the above theorem, states that: *If a meromorphic function f (or equivalently a holomorphic mapping $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$)omits three distinct points in $\mathbb{C} \cup \{\infty\}$, then f must be constant.* In 1929, by introducing the functions $T_f(r), N_f(r, \infty)$ and $m_f(r, \infty)$ (See below for definitions), R. Nevanlinna gave a quantitative version of the little Picard Theorem by establishing the so-called *Second Main Theorem for Meromorphic Functions.* It was then extended by H. Cartan and L. Ahlfors to case of meromorphic maps from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting hyperplanes, and recently by Min Ru to the hypersurface case. The above mentioned Nevanlinna theory generally works for meromorphic maps either on a parabolic-type complex manifolds, or on the ball $B(R) \subset \mathbb{C}^m$ with the growth condition $\lim_{r \longrightarrow R} \sup \dfrac{T_f(r, r_0)}{\log 1/(R-r)} = \infty$, where $T_f(r, r_0)$ is the growth function of f. Example of the value distribution of the Gauss map of a complete minimal surface on the disc $\Delta(R)$ shows that this growth condition isn't always satisfied i.e, $\lim_{r \longrightarrow R} \frac{T_f(r,r_0)}{\log 1/(R-r)}$ may be finite. To deal with this case, a new theory is needed. In this dissertation we develop a Nevanlinna theory for meromorphic mappings on a (non-parabolic-type) Kähler manifold M with a complete metric into $\mathbb{P}^n(\mathbb{C})$. In order to develop a Nevanlinna theory for the above mentioned Kähler manifold, we assume the following growth condition for f: there exists a nonzero bounded continuous real-valued function h on M such that $\rho \Omega_f + \dfrac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h^2 \geq \mathrm{Ric}\, \omega$ for some non-negative constant ρ, where Ω_f is the pull-back of the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$ and Ric ω is the Ricci form of the Kähler form ω. Using this new theory, we were able to give an extension (see Theorem 2.1.6) of the defect relation for hypersurfaces given by M. Ru in [16]. We also gave an application of our extension result to the Gauss map of a complete regular submanifold of $\mathbb{C}^m$ (see Theorem 2.3.2).

Another remarkable result of Nevanlinna is the following identity result: *If $a_1, ..., a_5$ are distinct points of the Riemann sphere, and f, g are non-constant meromorphic functions with $f^{-1}(a_i) = g^{-1}(a_i)$ for each $i = 1, ..., 5$, then $f \equiv g$.* In the higher dimensional generalization of Nevanlinna theory, the problem of extending these results has attracted attention; most of which involve intersections of holomorphic maps with hyperplanes in projective space. For example H. Fujimoto gave an extension of the uniqueness theorem to meromorphic mappings of a complete kähler manifold whose universal covering is biholomorphic to the ball in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which intersect hyperplanes (cf. [11]). In this thesis, we generalized Fujimoto's result to the case where the map intersects with hypersurfaces instead hyperplanes (see Theorem 2.2.2).

In Chapter 1, we review the relevent topics in complex analysis and geometry that are fundamental to our approach. In Chapter 2, we consider meromorphic maps on M into projective space. We will prove a non-integrated defect relation for meromorphic maps of complete Kähler manifold under the assumption that the universal covering is biholomorphic to a ball in $\mathbb{C}^m$. We then give a uniqueness theorem of meromorphic maps on M into projective space. We end the chapter with the non-integrated defect for the special case of the Gauss map of a complete regular submanifold of $\mathbb{C}^m$.

1.2 Nevanlinna theory of meromorphic functions on the disc $\Delta(R) \subseteq \mathbb{C}$

In this section we recall the fundamental ideas of Nevanlinna theory of meromorphic functions. We consider holomorphic maps $f : \mathbb{C} \longrightarrow M$, where $\mathbb{C}$ is the complex number plane and M is a complex manifold. In particular, it asks when is f forced to be constant. An example is taking M as a disc of finite radius. In this case, the classical Liouville's theorem says that every holomorphic map sending $\mathbb{C}$ into the disc must be constant. In this chapter, we consider the case where $M = \mathbb{P}^1(\mathbb{C})$, the complex projective space of dimension 1.

The fundamental tool of this subject is the measurement of the growth of the map f. Given an entire function, there are two ways of measuring its rate of growth-its maximum modulus on the disc of radius r (viewed as a function of r) and the maximum number of times it takes a value in the image on the disc. Unfortunately, the maximum modulus doesn't work for meromorphic function since it may become infinite at some finite values of r. R. Nevanlinna found the right substitute for the maximum modulus by introducing the characteristic function $T_f(r)$ to measure the growth of f (see [1]). In this section, we'll recall Nevanlinna's *First* and *Second Main Theorem* for meromorphic functions with application to the uniqueness problem.

1.2.1 The first main theorem

We begin by recalling the following well-known Poisson-Jensen formula in classical complex analysis. The proof can be found in [17].

Theorem 1.2.1 *(Poisson-Jensen Formula) Let $f \not\equiv 0$ be a meromorphic function on the closed disc $\overline{D}(R)$, $R < \infty$. Let $a_1, ..., a_q$ denote the zeros of f in $\overline{D}(R)$, counting multiplicities, and $b_1, ..., b_q$ denote the poles of f in $\overline{D}(R)$, also counting multiplicities. Then for any z, with $|z| < R$, which is not a zero or a pole, we have*

$$\log|f(z)| = \int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi}$$
$$- \sum_{i=1}^{p} \log\Big|\frac{R^2 - \overline{a}_i z}{R(z - a_i)}\Big| + \sum_{j=1}^{q} \log\Big|\frac{R^2 - \overline{b}_j z}{R(z - b_j)}\Big|.$$

Let $z_0 \in \overline{D}(R)$. If $f(z) = c(z - z_0)^m + ...$, where c is the leading nonzero coefficient, then m is called the order of f at z_0 and is denoted by $\mathrm{ord}_{z_0} f$.

Corollary 1.2.2 *(Jensen Formula) Let $f \not\equiv 0$ be a meromorphic function on the closed disc $\overline{D}(R)$, $R < \infty$. Let $a_1, ..., a_q$ denote the zeros of f in $\overline{D}(R) - \{0\}$, counting multiplicities, and $b_1, ..., b_q$ denote the poles of f in $\overline{D}(R) - \{0\}$, also counting multiplicities. Then*

$$\log|c_f| = \int_0^{2\pi} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi} - \sum_{i=1}^{p} \log\Big|\frac{R}{a_i}\Big| + \sum_{j=1}^{q} \log\Big|\frac{R}{b_j}\Big| - (ord_0 f)\log R,$$

where $f(z) = c_f z^{ord_0 f} + ...$, $ord_0 f \in \mathbb{Z}$, *and* c_f *is the leading nonzero coefficient.*

Proof: Consider the function $f z^{-\mathrm{ord}_0 f}$. Then applying theorem 1.2.1 to it at $z = 0$ gives the corollary. $\square$

We now proceed to define Nevanlinna functions. Let f be a meromorphic function on the closed disc $\overline{D}(R)$, $0 < R \leq \infty$ and let $0 < r < R$. Denote the number of poles of f on the closed disc $\overline{D}(R)$ by $n_f(r, \infty)$, counting multiplicity. We define the *counting function* $N_f(r, \infty)$ to be

$$N_f(r, \infty) = n_f(0, \infty) \log(r) + \int_0^r \frac{n_f(t, \infty) - n_f(0, \infty)}{t} dt,$$

where $n_f(0, \infty)$ is the multiplicity if f has a pole at $z = 0$. If $a \in \mathbb{C}$, define the counting function with respect to a by

$$N_f(r, a) = N_{\frac{1}{f-a}}(r, \infty).$$

By the definition of the Lebesgue-Stieltjes integral, we have

$$N_f(r, 0) = (\mathrm{ord}_0^+ f) \log r + \sum_{z \in D(r) z \neq 0} (\mathrm{ord}_z^+ f) \log \left| \frac{r}{z} \right|.$$

where $ord_z^+ f = \max\{0, ord_z f\}$ is the multiplicity of the zero at z. We note that $N_f(r, a)$ measures how many times f takes the value a. Define the *proximity function* $m_f(r, \infty)$ by

$$m_f(r, \infty) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi},$$

where $\log^+(x) = \max(0, \log x)$. For a complex number a, we define $m_f(r, a) = m_{\frac{1}{f-a}}(r, \infty)$. The proximity function measures, on average, how close f is to a in the disk of radius r. Finally, we let

$$T_f(r) = N_f(r, \infty) + m_f(r, \infty).$$

The function $T_f(r)$ is called *Nevanlinna's characteristic function.* The fundamental idea in Nevanlinna theory is that the growth of the characteristic function carries considerable information about the meromorphic function f. Thus, producing bounds for $T_f(r)$ can be translated into statements about f. For example, $T_f(r) = O(1)$ if and only if f is a constant, and $T_f(r) = O(\log(r))$ if and only if f is a rational function. Consequently, one must develop techniques for estimating the characteristic function. Nevanlinna's main results show that in fact $T_f(r)$ can be estimated using the counting and proximity functions. We now state the First Main Theorem.

Theorem 1.2.3 *(first main theorem) Suppose $f \not\equiv 0$ be a meromorphic function on the closed disk $\overline{D(R)}$, $R \leq \infty$. Then, for any $0 \leq r < R$,*

$$(i)\ T_f(r) = N_f(r, 0) + m_f(r, 0) + \log|c_f|.$$

Given a complex number a,

$$(ii)\ |T_f(r) - m_f(r, a) - N_f(r, a)| \leq |\log|c_{1/(f-a)}| + \log^+|a| + \log 2|,$$

where $c_{1/(f-a)}$ is the leading nonzero coefficient in the Taylor expansion of $1/(f-a)$ around 0.

Proof: First note that Jensen Formula can be rewritten as

$$\begin{aligned}
\log|c_f| &= \int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} - \sum_{z \in D(r), z \neq 0} (\mathrm{ord}_z f) \log\left|\frac{r}{z}\right| - (\mathrm{ord}_0 f) \log r \\
&= \int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} + N_f(r, \infty) - N_f(r, 0) \\
&= \int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} + T_f(r) - m_f(r, \infty) - N_f(r, 0). \qquad (1.1)
\end{aligned}$$

Now if $\log|f(re^{i\theta})| > 0$, then $f > 1$, $1/f < 1$ and

$\int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} - m_f(r,\infty) = 0 = m_f(r,0)$.

If $\log|f(re^{i\theta})| < 0$, then $f < 1,\ 1/f > 1,\ m_f(r,\infty) = 0$ and $m_f(r,0) = -\int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi}$. Plug in these into (1.1) gives (i).

To prove (ii), we consider the function $1/(f-a)$ and apply Jensen Formula to it to get:

$$\log|c_{1/(f-a)}| = \int_0^{2\pi} \log\frac{1}{|f(re^{i\theta}) - a|}\frac{d\theta}{2\pi} + N_{1/(f-a)}(r,\infty) - N_{(f-a)}(r,0).$$

Since $\log x = \log^+ x - \log^+(1/x)$, we have that

$$\log|c_{1/(f-a)}| = \int_0^{2\pi} \log^+\frac{1}{|f(re^{i\theta}) - a|}\frac{d\theta}{2\pi} - \int_0^{2\pi} \log^+|f(re^{i\theta}) - a|\frac{d\theta}{2\pi} + N_f(r,a) - N_f(r,\infty).$$

Thus

$$\int_0^{2\pi} \log^+|f(re^{i\theta}) - a|\frac{d\theta}{2\pi} = m_f(r,a) - N_f(r,\infty) + N_f(r,a) - \log|c_{1/(f-a)}|.$$

Now, we note that for positive numbers x and y, we have

$$\log^+|(x+y)| \leq \log^+ 2\max\{x,y\} \leq \log^+|x| + \log^+|y| + \log 2.$$

So

$$|\log^+|(x+y)| - \log^+|x|| \leq \log^+|y| + \log 2.$$

Therefore

$$|T_f(r) - m_f(r,a) - N_f(r,a) + \log|c_{1/(f-a)}|| \leq \log^+ a + \log 2.$$

and (ii) follows. $\square$

The *First Main Theorem* states that $T_f(r) = m_f(r,a) + N_f(r,a) + O(1)$. It gives an upper bound for $N_f(r,a)$ in terms of $T_f(r)$, hence on the number of times f takes on the value a. Our next goal is to prove the *Second Main Theorem* which will provide a lower bound for $N_f(r,a)$ in terms of $T_f(r)$. To do so, we first introduce the "logarithmic derivative lemma“.

1.2.2 The logarithmic derivative lemma

This section is concerned with the logarithmic derivative lemma as presented in [17].

Theorem 1.2.4 *(Gol'dberg-Grinshtein estimate)*

Let f be a meromorphic function on $D(R)$, $0 < R \leq \infty$, and let $0 < \alpha < 1$, then , for $r_0 < r < \rho < R$, we have

$$\int_0^{2\pi} \Big|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\Big|^{\alpha} \frac{d\theta}{2\pi} \leq 2^{\alpha}\Big(\frac{\rho}{r(\rho - r)}\Big)^{\alpha}(2T_f(\rho) - \log|c_f|)^{\alpha}$$
$$+2^{\alpha+4}\sec(\alpha\pi/2)\Big(\frac{\rho}{r(\rho - r)}\Big)^{\alpha}\Big(T_f(\rho) + |\mathrm{ord}_0 f|\log^+\frac{1}{r_0}\Big)^{\alpha},$$

where $f(z) = c_f z^{ord_0 f} + \ldots$, $ord_0 f \in \mathbb{Z}$, and c_f is the leading nonzero coefficient.

Lemma 1.2.5 *(Borel's growth lemma)*

Let $F(r)$ be a positive, non decreasing, continuous function defined on $[r_0, \infty)$ with $r_0 \geq e$ such that $F(r) \geq e$ on $[r_0, \infty)$. Then, for every $\epsilon > 0$, there exists a closed set $E \subset [r_0, \infty)$ of finite Lebesgue measure such that if we set $\rho = r + \frac{1}{\log^{1+\epsilon} F(r)}$ for all $r \geq r_0$ and not in E, we have

$$\log F(\rho) \leq \log F(r) + 1$$

and

$$\log^+ \frac{\rho}{r(\rho - r)} \leq (1+\epsilon)\log^+\log F(r) + \log 2.$$

Theorem 1.2.6 *(lemma on the logarithmic derivative)*

Let f be a non-constant meromorphic function on $\mathbb{C}$. Assume that $T_f(r_0) \geq e$ for some r_0. Then, for any $\epsilon > 0$, the inequality

$$m_{f'/f}(r, \infty) \leq \log T_f(r) + (1+\epsilon) \log^+ T_f(r) + C$$

holds for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ with finite Lebesgue measure, where C is a constant which depends only on f.

Proof: Using the concavity of the $\log^+$ function, we have for $\alpha > 0$

$$m_{f'/f}(r, \infty) = \frac{1}{\alpha} \int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^\alpha \frac{d\theta}{2\pi} \leq \frac{1}{\alpha} \log^+ \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^\alpha \frac{d\theta}{2\pi}. \tag{1.2}$$

We now apply Lemma 1.2.5 with $\rho = r + \frac{1}{\log^{1+\epsilon} T_f(r)}$ to the function $F(r) = T_f(r)$ to get that, for every $r \geq r_0$ not in the set E,

$$\log T_f(\rho) \leq \log T_f(r) + 1$$

and

$$\log^+ \frac{\rho}{r(\rho - r)} \leq (1+\epsilon) \log^+ \log T_f(r) + \log 2.$$

By theorem 1.2.4, we have

$$\begin{aligned}
m_{f'/f}(r,\infty) \leq{}& \frac{1}{\alpha}\log^+\Big\{2^\alpha\Big(\frac{\rho}{r(\rho-r)}\Big)^\alpha(2T_f(\rho)-\log|c_f|)^\alpha \\
&+2^{\alpha+4}\sec(\alpha\pi/2)\Big(\frac{\rho}{r(\rho-r)}\Big)^\alpha\Big(T_f(\rho)+|\mathrm{ord}_0 f|\log^+\frac{1}{r_0}\Big)^\alpha\Big\} \\
&\leq \frac{1}{\alpha}\log^+\Big(\frac{\rho}{r(\rho-r)}\Big)^\alpha+ \\
&\frac{1}{\alpha}\log^+\Big\{2^\alpha(2T_f(\rho)-\log|c_f|)^\alpha+ \\
&2^{\alpha+4}\sec(\alpha\pi/2)\Big(T_f(\rho)+|\mathrm{ord}_0 f|\log^+\frac{1}{r_0}\Big)^\alpha\Big\}+\log 2 \\
&\leq \log^+\frac{\rho}{r(\rho-r)}+\log^+T_f(\rho)+C \\
&\leq (1+\epsilon)\log^+\log T_f(r)+\log^+T_f(r)+C.
\end{aligned}$$

The theorem follows since $r \geq r_0$ and $T_f(r_0) \geq e$. □

1.2.3 The second main theorem

For a meromorphic function f , we define a ramification term $N_{ram(f)}(r) = N_{f'}(r,0) + 2N_f(r,\infty) - N_{f'}(r,\infty)$. Using this, we now introduce a much more subtle and powerful estimate for $T_f(r)$.

Theorem 1.2.7 *(The second main theorem) Suppose $a_1, ..., a_q$ are distinct complex numbers, and f is a non-constant meromorphic function on $B(R) \subseteq \mathbb{C}$; $0 < R \leq \infty$. Then for every $\epsilon > 0$, the inequality*

$$\begin{aligned}
&(q-1)T_f(r)+N_{ram(f)}(r) \\
&\leq \sum_{j=1}^{q} N_f(r,a_j)+N_f(r,\infty)+\log T_f(r)+(1+\epsilon)\log^+\log T_f(r)+O(1),
\end{aligned}$$

holds for $r \geq r_0$ outside of a set $E \subset (0,+\infty)$ of finite Lebesgue measure.

Proof: Let $\delta = \min_{i \neq j}\{|a_i - a_j|, 1\}$. For each z with $f(z) \neq \infty$ and $f(z) \neq a_j$ for $1 \leq j \leq q$, let j_0 be the index among $\{1, 2, ..., q\}$, such that

$$|f(z) - a_{j_0}| \leq |f(z) - a_j| \ for\ all\ 1 \leq j \leq q.$$

Then for $j \neq j_0$, by the triangle inequality, $|f(z) - a_j| \geq \delta/2$. Thus, for $j \neq j_o$,

$$\begin{aligned}\log^+ |f(z)| &\leq \log^+ |f(z) - a_j| + \log^+ |a_j| + \log 2 \\ &= \log |f(z) - a_j| + \log^+ \frac{1}{|f(z) - a_j|} + \log^+ |a_j| + \log 2 \\ &\leq \log |f(z) - a_j| + \log^+ \delta/2 + \log^+ |a_j| + \log 2.\end{aligned}$$

Therefore

$$(q-1)\log^+ |f(z)| \leq \sum_{j \neq j_0} \log |f(z) - a_j| + \sum_{j=1}^{q} \log^+ |a_j| + (q-1)(\log^+ \frac{2}{\delta} \log 2).$$

Now

$$\begin{aligned}\sum_{j \neq j_0} \log |f(z) - a_j| &= \sum_{j=1}^{q} \log |f(z) - a_j| - \log |f'(z)| + \log \frac{|f'(z)|}{|f(z) - a_{j_0}|} \\ &\leq \sum_{j=1}^{q} \log |f(z) - a_j| - \log |f'(z)| + \log \Big(\sum_{j=1}^{q} \frac{|f'(z)|}{|f(z) - a_j|} \Big).\end{aligned}$$

Thus

$$(q-1)\log^+ |f(z)| \leq \sum_{j=1}^{q} \log |f(z) - a_j| - \log |f'(z)| + \log \Big(\sum_{j=1}^{q} \frac{|f'(z)|}{|f(z) - a_j|} \Big) + \\ \sum_{j=1}^{q} \log^+ |a_j| + (q-1)(\log^+ \frac{2}{\delta} + \log 2).$$

Now, we set $z = re^{i\theta}$ and integrate with respect to θ to get

$$(q-1)m_f(r, \infty) \leq \sum_{j=1}^{q} \int_0^{2\pi} \log |f(re^{i\theta}) - a_j| \frac{d\theta}{2\pi} - \int_0^{2\pi} \log |f'(re^{i\theta})| \frac{d\theta}{2\pi} + \\ \int_0^{2\pi} \log \Big(\sum_{j=1}^{q} \frac{|f'(re^{i\theta})|}{|f(re^{i\theta}) - a_j|} \Big) \frac{d\theta}{2\pi} + O(1).$$

From Jensen formula, we have

$$\int_0^{2\pi} \log |f(re^{i\theta}) - a_j| \frac{d\theta}{2\pi} = N_f(r, a_j) - N_f(r, \infty) + \log |c_{f-a_j}|,$$

and

$$\int_0^{2\pi} \log |f'(re^{i\theta})| \frac{d\theta}{2\pi} = N_{f'}(r, 0) - N_{f'}(r, \infty) + \log |c_{f'}|.$$

Thus, the last inequality above becomes

$$\begin{aligned}
&(q-1)m_f(r,\infty) - \sum_{j=1}^{q} N_f(r, a_j) + qN_f(r,\infty) + N_{f'}(r,0) - N_{f'}(r,\infty) \\
&\leq \int_0^{2\pi} \log \Big(\sum_{j=1}^{q} \frac{|f'(re^{i\theta})|}{|f(re^{i\theta}) - a_j|} \Big) \frac{d\theta}{2\pi} + \sum_{j=1}^{q} \log^+ |a_j| + (q-1)(\log^+ \frac{2}{\delta} + \log 2) + \\
&\sum_{j=1}^{q} \log |c_{f-a_j}| - \log |c_{f'}|. \qquad (1.3)
\end{aligned}$$

By the first main theorem and the definition of $N_{\mathrm{ram},f}(r)$, the left hand side of (1.3) is

$$(q-1)T_f(r) + N_{\mathrm{ram},f}(r) - \sum_{j=1}^{q} N_f(r, a_j) - N_f(r, \infty).$$

To complete the proof, we'll now estimate

$$\int_0^{2\pi} \log \Big(\sum_{j=1}^{q} \frac{|f'(re^{i\theta})|}{|f(re^{i\theta}) - a_j|} \Big) \frac{d\theta}{2\pi}.$$

Let α be a real number between 0 and 1. Then

$$\begin{aligned}
\int_0^{2\pi} \log \Big(\sum_{j=1}^{q} \frac{|f'(re^{i\theta})|}{|f(re^{i\theta}) - a_j|} \Big) \frac{d\theta}{2\pi} &= \frac{1}{\alpha} \int_0^{2\pi} \log \Big(\sum_{j=1}^{q} \frac{|f'(re^{i\theta})|}{|f(re^{i\theta}) - a_j|} \Big)^{\alpha} \frac{d\theta}{2\pi} \\
&\leq \frac{1}{\alpha} \int_0^{2\pi} \log \Big(\sum_{j=1}^{q} \Big| \frac{f'(re^{i\theta})}{f(re^{i\theta}) - a_j} \Big|^{\alpha} \Big) \frac{d\theta}{2\pi} \\
&\leq \frac{1}{\alpha} \log \Big(\sum_{j=1}^{q} \int_0^{2\pi} \Big| \frac{f'(re^{i\theta})}{f(re^{i\theta}) - a_j} \Big|^{\alpha} \Big) \frac{d\theta}{2\pi}
\end{aligned}$$

where we have used the concavity of the logarithm and the inequality $(\sum_j a_j)^{\alpha} \leq \sum_j a_j^{\alpha}$ for positive numbers a_j and $\alpha, 1 < \alpha < 1$.

By theorem 1.2.4 and using the fact that $\log^+(x+y) \leq \log^+ x + \log^+ y$, we have

$$\begin{aligned}
&\log\Big(\sum_{j=1}^{q}\int_0^{2\pi}\Big|\frac{f'(re^{i\theta})}{f(re^{i\theta})-a_j}\Big|^{\alpha}\frac{d\theta}{2\pi}\Big) \\
&\leq \log\sum_{j=1}^{q}\Big\{2^{\alpha}\Big(\frac{\rho}{r(\rho-r)}\Big)^{\alpha}(2T_{f-a_j}(\rho)-\log|c_{f-a_j}|)^{\alpha} \\
&+2^{\alpha+3}\sec(\alpha\pi/2)\Big(\frac{\rho}{r(\rho-r)}\Big)^{\alpha}\Big(2T_{f-a_j}(\rho)+2|\mathrm{ord}_0(f-a_j)|\log^+\frac{1}{r_0}\Big)^{\alpha}\Big\} \\
&\leq \alpha\log\Big(\frac{\rho}{r(\rho-r)}\Big)+\alpha\log\sum_{j=1}^{q}2T_{f-a_j}(\rho)+\alpha C(\alpha) \\
&\leq \alpha\log^+\Big(\frac{\rho}{r(\rho-r)}\Big)+\alpha\log^+\sum_{j=1}^{q}2T_{f-a_j}(\rho)+\alpha C(\alpha),
\end{aligned}$$

where $C(\alpha)$ is a constant depending on α. We now apply Lemma 1.2.5 with $\rho = r + \frac{1}{\log^{1+\epsilon}T_f(r)}$ to the function $F(r) = T_f(r)$ to get that, for every $r \geq r_0$ not in the set E,

$$\log T_f(\rho) \leq \log T_f(r) + 1$$

and

$$\log^+\frac{\rho}{r(\rho-r)} \leq (1+\epsilon)\log^+\log T_f(r) + \log 2.$$

It then follows that for $r \geq r_0$ not in E,

$$\begin{aligned}
&\log^+\Big(\frac{\rho}{r(\rho-r)}\Big)+\log^+\sum_{j=1}^{q}2T_{f-a_j}(\rho)+C(\alpha) \\
&\leq (1+\epsilon)\log^+\log T_f(r)+\log^+\max_{1\leq j\leq q}\{2T_{f-a_j}(\rho)\}+C(\alpha) \\
&\leq (1+\epsilon)\log^+\log T_f(r)+\log 2T_f(\rho)+C(\alpha) \\
&\leq (1+\epsilon)\log^+\log T_f(r)+\log T_f(\rho)+C(\alpha) \\
&\leq (1+\epsilon)\log^+\log T_f(r)+\log T_f(r)+C(\alpha).
\end{aligned}$$

Therefore (1.3) becomes

$$(q-1)T_f(r) + N_{\mathrm{ram},f}(r) - \sum_{j=1}^{q} N_f(r,a_j) - N_f(r,\infty) \leq$$

$$(1+\epsilon)\log^+ \log T_f(r) + \log T_f(r) + O(1).$$

The proof of the theorem is therefore completed. $\square$

An appropriate version of the second main theorem will be used for our proves. In this situation it is necessary to have a version of this theorem which involves the so-called *truncated counting functions.* We define $n_f^{(k)}(r,\infty)$ by counting all poles with multiplicity greater than k as pole with multiplicity k. We define $N_f^{(k)}(r,\infty)$ accordingly. Then the second main theorem can be used to deduce the following inequality:

Theorem 1.2.8 *(second main theorem with truncation) let $a_1, ..., a_q$ be complex numbers and f be as in theorem 1.2.7, then for every $\epsilon > 0$, the inequality*

$$(q-1)T_f(r) \leq \sum_{j=1}^{q} N_f^{(1)}(r,a_j) + N_f^{(1)}(r,\infty) +$$

$$(1+\epsilon)\log^+ \log T_f(r) + \log T_f(r) + O(1),$$

holds for all $r \geq r_0$ outside a set $E \subset (0,\infty)$ with finite lebesgue measure.

Proof: We note first that

$$\sum_{j=1}^{q} N_f(r,a_j) + N_f(r,\infty) - N_{\mathrm{ram},f}(r) \leq \sum_{j=1}^{q} N_f^{(1)}(r,a_j) + N_f^{(1)}(r,\infty).$$

From this theorem 1.2.8 follows. $\square$

Corollary 1.2.9 *(Picard's theorem) If a meromorphic function f on $\mathbb{C}$ omits three distinct points $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$, then f must be constant.*

Proof: We'll prove that if f is not constant, then we have a contradiction. Indeed, if f is assumed nonconstant, then by theorem 1.2.7, we have

$$3T_f(r) - \sum_{j=1}^{3} N_f(r, a_j) + N_{\text{ram},f}(r) \leq T_f(r) + N_f(r, \infty) +$$
$$(1+\epsilon)\log^+ \log T_f(r) + \log T_f(r) + O(1).$$

Using the fact that $N_{\text{ram},f}(r) \geq 0$ and $N_f(r, \infty) \leq T_f(r)$, we have

$$\sum_{j=1}^{3} m_f(r, a_j) \leq 2T_f(r) + (1+\epsilon)\log^+ \log T_f(r) + \log T_f(r) + O(1)$$

holds outside a set $E \subset (0, +\infty)$ of finite Lebesgue measure. However since f omits the points a_js, we have that $m_f(r, a_j) = T_f(r) + O(1)$. Thus

$$3T_f(r) \leq 2T_f(r) + (1+\epsilon)\log^+ \log T_f(r) + \log T_f(r) + O(1),$$

which is a contradiction since $\log x < x$ for positve number x. $\square$

We now recall from elementary complex analysis the identity theorem.

Theorem 1.2.10 *Suppose $f, g : D \longrightarrow \mathbb{C}$ are holomorphic on a domain $D \subset \mathbb{C}^m$, and $f = g$ on some set $Z \subset D$ with a limit point in D. Then, $f \equiv g$.*

Proof: Set $h = f - g$. We must show that $h \equiv 0$. Let $A = \{z \in D | h^{(n)}(z) = 0 \text{ for all } n\}$. We show that A is non-empty. Let a be a limit point of Z. Then there is a sequence $\{z_n\} \subset Z$ so that $z_n \neq a,\ z_n \to a$, and $h(z_n) = 0$ for all n. Write

$$h(z) = \sum_{j=0}^{\infty} c_j (z-a)^j.$$

Then, we have that $-c_0 = \sum_{j=1}^{\infty} c_j (z_n - a)^j \to 0$ as $n \to \infty$. Hence, $c_0 = 0$. Suppose we have that $c_0 = ... = c_k = 0$. Then we can write

$$0 = h(z_n) = (z_n - a)^{k+1}(c_{k+1} + c_{k+2}(z_n - a)...).$$

Since $z_n - a$ is not zero, by passing to the limit again we see that $c_{k+1} = 0$. Thus, by induction all coefficients vanish, hence all derivatives at a vanish, proving A is non-empty. We note that by continuity, A is closed. If $z_0 \in A$, then in a neighborhood of z_0 we can write

$$h(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j.$$

We then have all $a_j = 0$ since $z_0 \in A$; hence, the entire neighborhood is contained in A, which shows that A is open. By connectedness h must vanish identically on D. $\square$

To prove our uniqueness result our approach shall be to prove that the two functions in question f and g say, agree locally. In the higher dimensional setting this is still sufficient to conclude that $f \equiv g$ by vertue of the following identity theorem.

Theorem 1.2.11 *(Identity theorem) Suppose $f, g : S \longrightarrow M$ are holomorphic, where S is a connected Riemann surface and M a complex $n-$manifold. If there is an open set $U \subset S$ so that $f = g$ on U then $f \equiv g$.*

Proof:

First we note that if the condition is satisfied for $f, g : D \longrightarrow \mathbb{C}^m$, where $D \subset \mathbb{C}$ is a domain, then $f \equiv g$ by the previous theorem (since they are equal componentwise). Now, let R denote the subset of S consisting of those points which have a neighborhood on which $f = g$. By assumption R is non-empty and it is clear that R is open. We further claim that R is closed. Suppose p is a point in closure of R. By continuity $f(p) = g(p)$. Let $V \subset S$ and $W \subset M$ be connected coordinate neighborhoods with $f(V)$ and $g(V)$ contained in W. By composing with coordinate charts we may view $f, g : V \subset \mathbb{C} \longrightarrow W \subset \mathbb{C}^m$. Then since $V' = V \cap R$ is open and non-empty, $f = g$ on V' and V is connected. It then follows that

$f = g$ on V. Thus $p \in R$ and hence, R is closed. By hypothesis, R is non-empty, hence $R = S$ since S is connected. Thus $f \equiv g$. $\square$

Remark 1.2.12 *In fact, the statement above is true if S is any connected complex manifold, not merely a Riemann surface.*

We now use the above machinery to give the proof to Nevanlinna five points uniqueness theorem.

Theorem 1.2.13 *(Nevanlinna five points uniqueness theorem) If $a_1, ..., a_5$ are distinct points of the Riemann sphere, and f, g are non-constant meromorphic functions with $f^{-1}(a_i) = g^{-1}(a_i)$ for each $i = 1, ..., 5$, then $f \equiv g$.*

Proof: Suppose $f^{-1}(a_j) = g^{-1}(a_j)$ for $j = 1, ..., 5$, but f is not identical to g. By the Second Main Theorem with truncation, we have:

$$3T_f(r) \leq \sum_{j=1}^{5} N_f^{(1)}(r, a_j) + O(\log T_f(r)),$$

$$3T_g(r) \leq \sum_{j=1}^{5} N_g^{(1)}(r, a_j) + O(\log T_g(r)).$$

Adding these two inequalities we obtain:

$$3(T_f(r) + T_g(r)) \leq \sum_{j=1}^{5} (N_f^{(1)}(r, a_j) + N_g^{(1)}(r, a_j)) + O(\log T_f(r) + \log T_g(r)).$$

Since $f^{-1}(a_j) = g^{-1}(a_j)$, we obtain $\sum_{j=1}^{5} N_f^{(1)}(r, a_j) \leq N_{f-g}(r, 0)$ and similarly for g. Hence,

$$3(T_f(r) + T_g(r)) \leq 2N_{f-g}(r, 0) + O(\log T_f(r) + \log T_g(r)).$$

But $N_{f-g}(r,0) \leq T_{f-g}(r) + O(1) \leq T_f(r) + T_g(r) + O(1)$, and so the inequality above implies

$$T_f(r) + T_g(r) \leq O(\log T_f(r) + \log T_g(r)),$$

which is a contradiction since f and g are not constant. Thus $f \equiv g$. $\square$

1.3 Nevanlinna theory of meromorphic maps on the ball $B(R) \subseteq \mathbb{C}^m$ into complex projective spaces

In this section, we will introduce Nevanlinna theory for meromorphic mappings on the ball $B(R) \subseteq \mathbb{C}^m,\ 0 < R \leq \infty$. There are two approaches in extending Nevanlinna theory for holomorphic curves in $\mathbb{P}^n(\mathbb{C})$. One is given by H. Cartan and the other is given by Ahlfors. We'll follow Cartan's approach which uses the logarithmic derivative lemma.

By a divisor on a domain G in $\mathbb{C}^m$ we mean a map ν of G into $\mathbb{Z}$ such that, for each $z_0 \in G$, there are nonzero holomorphic functions h and g on a connected neighborhood $U(\subset G)$ of z_0 so that $\nu(z) = \nu_h^0(z) - \nu_g^0(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m-2$.

Take a nonzero meromorphic function φ on a domain G in $\mathbb{C}^m$. For each $z_0 \in G$, we choose nonzero holomorphic functions g and h on a neighborhood $U(\subset G)$ of z_0 such that $\varphi = \frac{g}{h}$ on U and $\dim(f^{-1}(0) \cup g^{-1}(0)) \leq m-2$, we define $\nu_\varphi^\infty := \nu_h$, $\nu_\varphi^a := \nu_{g-ah}$ for $a \in \mathbb{C}$ and $\nu_\varphi = \nu_\varphi^0 - \nu_\varphi^\infty$, which are independent of the choices of h and g and so is globally well-defined on G. Let f be a meromorphic map of $B(R_0) \subseteq \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. We take holomorphic functions $f_0, f_1, \ldots, f_n$ such that $I_f := \{z \in B(R_0), f_0(z) = \cdots = f_n(z) = 0\}$

is of dimension at most $m-2$. Then, $f(z) = [f_0(z) : \cdots : f_n(z)]$ on $B(R_0) - I_f$ in terms of homogeneous coordinates $[w_0 : \cdots : w_n]$ on $\mathbb{P}^n(\mathbb{C})$ is called a reduced representation.

For $z = (z_1, ..., z_m) \in \mathbb{C}^m$ we set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ and define $B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}$, $\quad S(r) = \{z \in \mathbb{C}^m : \|z\| = r\}$ for $0 < r < +\infty$. Define

$$\sigma_m := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1} \qquad on \quad \mathbb{C}^m - \{0\},$$

$$v_l := (dd^c\|z\|^2)^l \qquad for \quad 1 \leq l \leq m.$$

Let $f(z) = [f_0(z) : \cdots : f_n(z)]$ be a reduced representation of f. Set $\|f\| := \left(|f_0|^2 + \cdots + |f_n|^2\right)^{1/2}$. Then the pullback of the normalized Fubini-Study metric form Ω on $\mathbb{P}^n(\mathbb{C})$ by f is given by

$$\Omega_f = dd^c \log\|f\|^2.$$

Fix $r_0 < R_0$, the characteristic function of f is defined by

$$T_f(r, r_0) = \int_{r_0}^{r} \frac{dt}{t^{2m-1}} \int_{B(t)} \Omega_f \wedge v_{m-1} \qquad (0 < r_0 < r < R_0).$$

We then have (see [23], p. 251-255),

$$T_f(r, r_0) = \int_{S(r)} \log\|f\|\sigma_m - \int_{S(r_0)} \log\|f\|\sigma_m.$$

Let μ_0 be a positive integer or ∞ and ν be a divisor on a domain $B(R_0) \subseteq \mathbb{C}^m$. Set $|\nu| = \overline{\{z \in B(R_0) : \nu(z) \neq 0\}}$. We define the counting function of ν truncated by μ_0 by

$$N_\nu^{[\mu_0]}(r_0, r) = \int_{r_0}^{r} \frac{n^{[\mu_0]}(t)}{t} dt$$

where

$$n^{[\mu_0]}(t) = \tfrac{1}{t^{2m-2}} \textstyle\int_{|\nu| \cap B(t)} \min\{\nu, \mu_0\} v_{m-1} \text{ if } m \geq 2,$$

$$n^{[\mu_0]}(t) = \sum_{|z|\leq t} \min\{\nu(z), \mu_0\} \qquad \text{if } m = 1.$$

Consider a hyperplane

$$H : a_0w_0 + ... + a_nw_n = 0$$

in $\mathbb{P}^n(\mathbb{C})$, where $A = (a_0, ..., a_n) \neq (0, ..., 0)$. Setting

$$\psi_f(H)(z) := \frac{\|f\|\|A\|}{|a_0f_0 + ... + a_nf_n|},$$

we define the proximity function of H by

$$m_f(r, H) := \int_{S(r)} \log |\psi_f(H)|\sigma_m - \int_{S(1)} \log |\psi_f(H)|\sigma_m.$$

The First Main Theorem is then stated as follows:

Theorem 1.3.1 *$T_f(r) = N(r, H) + m_f(r, H)$ for all hyperplanes H.*

1.3.1 The lemma of the logarithmic derivative and the generalized Wronskian

Let $\varphi(z_1, ..., z_m)$ be a nonzero meromorphic function on $B(R_0)$, $\quad 0 < R_0 \leq +\infty$. For a set $\alpha = (\alpha_1, ..., \alpha_m)$ of integers $\alpha_i \geq 0$ and $z = (z_1, ..., z_m) \in \mathbb{C}^m$, we set $\alpha! = \alpha_1!\alpha_2!...\alpha_m!$ $\quad |\alpha| = \alpha_1 + ... + \alpha_m \quad z^\alpha := z_1^{\alpha_1}...z_m^{\alpha_m}$ and $D^\alpha\varphi = D_1^{\alpha_1}...D_m^{\alpha_m}\varphi$, where $D_i\varphi = (\partial/\partial z_i)\varphi$.

The purpose of this section is to prove the following lemma of the logarithmic derivative.

Theorem 1.3.2 *(See [9],Theorem 3.1) Let ϕ be a meromorphic function on $B(R_0)$ and let $\alpha = (\alpha_1, ..., \alpha_m) \neq (0, ..., 0)$, $\quad 0 < r_0 < R_0$ and take positive numbers p, p' such that*

$0 < p|\alpha| < p' < 1$. *Then for* $r_0 < r < R < R_0$,

$$\int_{S(r)} \Big| z^\alpha (D^\alpha \phi/\phi)(z) \Big|^p \sigma_m(z) \leq K \Big(\frac{R^{2m-1}}{R-r} T_\phi(R, r_0) \Big)^{p'}.$$

Before proving this, we give the following Corollary to the Theorem, which is essentially the same as the lemma of logarithmic derivative in several variables given by Vitter (see [21]).

Corollary 1.3.3 *(See [9], Corollary 3.2) Let* $\alpha = (\alpha_1, ..., \alpha_m) \neq (0, ..., 0)$, *and* $0 < r_0 < R_0$. *For* $r_0 < r < R < R_0$ *we have,*

$$\int_{S(r)} \log^+ \Big| \Big(\frac{D^\alpha \varphi}{\varphi} \Big)(z) \Big| \sigma_m(z) \leq K \log^+ \Big(\frac{R^{2m-1}}{R-r} T_\varphi(R, r_0) \Big).$$

For the proof of the theorem, we recall some known facts.

Lemma 1.3.4 *([10], Lemma 2.5) Let* $r > 0$ *and* $0 < p < 1$. *For every* $a \in \mathbb{C}$, *we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{r^p}{|re^{i\theta} - a|^p} d\theta \leq \frac{2-p}{2(1-p)}.$$

For $z = (z_1, ..., z_m) \in \mathbb{C}^m$ set $\eta = (z_1, ..., z_{m-1})$, $\quad \zeta = z_m$, $\quad z = (\eta, \zeta)$ and $|\eta| = (|z_1|^2 + ... + |z_{m-1}|^2)^{1/2}$.

Lemma 1.3.5 *([3], P.35) Let* h *be an integrable function on* $S(r) \quad (r > 0)$. *Then*

$$\int_{S(r)} h\sigma_m = \frac{1}{r^{2m-2}} \int_{B(r)} v_{m-1}(\eta) \int_{|\zeta| = \sqrt{r^2 - |\eta|^2}} h(\eta, \zeta) \sigma_1(\zeta)$$

where $B(r) := \{\eta \in \mathbb{C}^{m-1} : |\eta| < r\}$.

For a nonzero meromorphic function φ on $B(R_0)$, there exists a subset E of $B(R_0)$ of measure zero such that for each $\eta \in B(R_0)\backslash E$ a meromorphic function $(\varphi|\eta)(\zeta) = \varphi(\eta, \zeta)$ is well defined on $\{\zeta \in \mathbb{C} : |\zeta| < \sqrt{R_0^2 - |\eta|^2}\}$.

Lemma 1.3.6 *([3], p.37) For each $a \in \mathbb{P}^1(\mathbb{C})$ and $0 < r < R_0$ we have*

$$\frac{1}{r^{2m-2}} \int_{B(r)\setminus E} n_{\nu^a_{\varphi|\eta}}(\sqrt{r^2 - |\eta|^2}) v_{m-1}(\eta) \leq n_{\nu^a_\varphi}(r).$$

We now prove the following:

Lemma 1.3.7 *([9], Lemma 3.8) Let $0 < \tilde{p} < 1$ and $0 < r < \rho < R_0$. For every $\eta \in B(r)\backslash E$, we have*

$$\int_{|\zeta|=\sqrt{r^2-|\eta|^2}} \Big|\zeta\Big(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\Big)(\eta, \zeta)\Big|^{\tilde{p}} \sigma_1(\zeta) \leq \Big(\frac{\rho}{\rho - r} \int_{|\zeta|=\sqrt{\rho^2-|\eta|^2}} |\log|\varphi(\eta, \zeta)||\sigma_1(\zeta)\Big)^{\tilde{p}}$$
$$+K(n_{\nu^0_\varphi}(\sqrt{\rho^2 - |\eta|^2}) + n_{\nu^\infty_\varphi}(\sqrt{\rho^2 - |\eta|^2})).$$

Proof: We may assume that $\varphi(\zeta) \neq 0$, on $\{\zeta : |\zeta| = \sqrt{\rho^2 - |\eta|^2}\}$, because each term is continuous in ρ. By differentiating the equation in Theorem 1.2.1 applied to the function $\varphi|\eta$ and $R = \tilde{\rho} := \sqrt{\rho^2 - |\eta|^2}$, we obtain

$$\Big(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\Big)(\eta, \zeta) = \frac{\tilde{\rho}}{\pi} \int_0^{2\pi} \frac{\log|\varphi(\eta, \zeta)|e^{i\phi}}{(\tilde{\rho}e^{i\phi} - \zeta)^2} d\phi - \sum_{|u|\leq\tilde{\rho}} \nu_{\varphi|\eta}(u)\Big\{\frac{1}{u - \zeta} - \frac{\bar{u}}{\tilde{\rho}^2 - \bar{u}\zeta}\Big\}.$$

Therefore,

$$\Big|\zeta\Big(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\Big)(\eta, \zeta)\Big|^{\tilde{p}} \leq \Big(2\tilde{\rho}|\zeta| \int_{|u|\leq\tilde{\rho}} \frac{|\log|\varphi(\eta, \zeta)||}{|u - \zeta|^2} \sigma_1(u)\Big)^{\tilde{p}} +$$
$$\sum_{|u|\leq\tilde{\rho}} (\nu^0_{\varphi|\eta}(u) + \nu^\infty_{\varphi|\eta}(u))\Big(\Big(\frac{|\zeta|}{|u - \zeta|}\Big)^{\tilde{p}} + \Big(\frac{|\zeta||u|}{|\tilde{\rho}^2 - \overline{u}\zeta|}\Big)^{\tilde{p}}\Big).$$

Integrating this and using Lemma 1.3.7, we see

$$
\begin{aligned}
&\int_{|\zeta|=\sqrt{r^2-|\eta|^2}} \Big|\zeta\Big(\frac{\partial\varphi}{\partial\zeta}\diagup\varphi\Big)(\eta,\zeta)\Big|^{\tilde p}\sigma_1(\zeta) \\
&\le \Big(2\tilde\rho\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}|\zeta|\sigma_1(\zeta)\int_{|u|=\tilde\rho}\frac{|\log|\varphi(\eta,\zeta)||}{|u-\zeta|^2}\sigma_1(u)\Big)^{\tilde p} \\
&\quad + \sum_{|u|\le\tilde\rho}(\nu^0_{\varphi|\eta}(u)+\nu^\infty_{\varphi|\eta}(u))\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}\Big(\frac{|\zeta|^{\tilde p}}{|u-\zeta|^{\tilde p}}+\frac{|\zeta|^{\tilde p}}{|(\tilde\rho^2\diagup u)\zeta|^{\tilde p}}\Big)\sigma_1(\zeta) \\
&\le \Big(2r\rho\int_{|u|\le\tilde\rho}|\log|\varphi(\eta,\zeta)||\sigma_1(u)\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}\frac{1}{|u-\zeta|^2}\sigma_1(\zeta)\Big)^{\tilde p}+ \\
&\quad K(\sum_{|u|\le\tilde\rho}(\nu^0_{\varphi|\eta}(u)+\nu^\infty_{\varphi|\eta}(u))).
\end{aligned}
$$

On the other hand, we have

$$\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}\frac{1}{|u-\zeta|^2}\sigma_1(\zeta)=\frac{1}{\tilde\rho^2-(r^2-|\eta|^2)}=\frac{1}{\rho^2-r^2}$$

for every u with $|u|=\tilde\rho$. From this we can conclude that

$$
\begin{aligned}
&\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}\Big|\zeta\frac{\partial\varphi}{\partial\zeta}(\eta,\zeta)\Big|^{\tilde p}\sigma_1(\zeta) \\
&\le \Big(\frac{2r\rho}{\rho^2-r^2}\int_{|u|=\tilde\rho}|\log|\varphi(\eta,\zeta)||\sigma_1(u)\Big)^{\tilde p}+K(n_{\nu^0_{\varphi|\eta}}(\tilde\rho)+n_{\nu^\infty_{\varphi|\eta}}(\tilde\rho)) \\
&\le \Big[2r\rho\Big(\frac{1}{2r(\rho-r)}-\frac{1}{2r(\rho+r)}\Big)\Big(\int_{|u|=\tilde\rho}|\log|\varphi(\eta,\zeta)||\sigma_1(u)\Big)\Big]^{\tilde p} \\
&+K(n_{\nu^0_{\varphi|\eta}}(\tilde\rho)+n_{\nu^\infty_{\varphi|\eta}}(\tilde\rho)) \\
&\le \Big(\frac{\rho}{\rho-r}\int_{|u|=\tilde\rho}|\log|\varphi(\eta,\zeta)||\sigma_1(u)\Big)^{\tilde p}+K(n_{\nu^0_{\varphi|\eta}}(\tilde\rho)+n_{\nu^\infty_{\varphi|\eta}}(\tilde\rho)).
\end{aligned}
$$

This concludes the proof. □

Proof of Theorem 1.3.2 for the case $|\alpha|=1$.

Proof: We prove the Theorem by induction on $|\alpha|$. We first consider the case $|\alpha|=1$. Without loss of generality, we may assume $D^\alpha=D_m$. Let $r_0<r<R<R_0,\quad 0<$

$p < p' < 1$ and set $\tilde{p} = p/p'$, $\rho = (R+r)/2$. Since each pole of $D_m\varphi \diagup \varphi$ is of order ≤ 1, $|z_m(D\varphi\diagup\varphi)(z)|^{\tilde{p}}$ is integrable on $S(r)$. By Lemma 1.3.4, 1.3.5, 1.3.6, and the Hölder inequality, we get

$$
\begin{aligned}
&\int_{S(r)} |z_m(D_m\varphi\diagup\varphi)(z)|^{\tilde{p}}\sigma_m(z)\\
&= \frac{1}{r^{2m-2}}\int_{|\eta|\leq r} v_{m-1}(\eta)\int_{|\zeta|=\sqrt{r^2-|\eta|^2}} |\zeta(D_m\varphi\diagup\varphi)(\eta,\zeta)|^{\tilde{p}}\sigma_1(\zeta)\\
&\leq \frac{1}{r^{2m-2}}\Big(\frac{\rho}{\rho-r}\Big)^{\tilde{p}}\Big(\int_{|\eta|\leq r} v_{m-1}(\eta)\Big)^{1-\tilde{p}}\\
&\times\Big(\int_{|\eta|\leq r} v_{m-1}(\eta)\int_{|\zeta|=\sqrt{\rho^2-|\eta|^2}} |\log|\varphi(\eta,\zeta)||\sigma_1(\zeta)\Big)^{\tilde{p}}\\
&+\frac{K}{r^{2m-2}}\int_{|\eta|\leq r}(n_{\nu^0_{\varphi|\eta}}(\sqrt{\rho^2-|\eta|^2})+n_{\nu^\infty_{\varphi|\eta}}(\sqrt{\rho^2-|\eta|^2}))v_{m-1}(\eta)\\
&\leq \Big(\frac{\rho}{\rho-r}\int_{S(\rho)}|\log|\varphi||\sigma_m\Big)^{\tilde{p}} + K\Big(\frac{\rho}{r}\Big)^{2m-2}(n_{\nu^0_{\varphi|\eta}}(\rho)+n_{\nu^\infty_{\varphi|\eta}}(\rho)).
\end{aligned}
$$

Moreover, using the fact that $\int_{S(r)}|\log|\varphi||\sigma_m \leq 2T_\varphi(r,r_0)+K\ \varphi$ and $n_{\nu^a_\varphi}(\rho) \leq \frac{2R}{R-r}(T_\varphi(R,r_0)+K)$ for a meromorphic map, we conclude that

$$
\begin{aligned}
&\int_{S(r)} |z_m(D_m\varphi\diagup\varphi)(z)|^{p}\sigma_m(z)\\
&\leq \Big(\int_{S(r)}|z_m(D_m\varphi\diagup\varphi)|^{\tilde{p}}\sigma_m(z)\Big)^{p'}\\
&\leq \Big(\frac{\rho}{\rho-r}\int_{S(r)}|\log|\varphi||\sigma_m\Big)^{\tilde{p}p'} + K\Big(\frac{\rho}{r}\Big)^{2m-2}(n_{\nu^0_\varphi}(\rho)^{p'}+n_{\nu^\infty_\varphi}(\rho)^{p'})\\
&\leq \Big(\frac{2R}{R-r}\int_{S(\rho)}|\log|\varphi||\sigma_m\Big)^{p} + K\Big(\frac{4R^{2m-1}}{R-r}(T_\varphi(R,r_0)+K)\Big)^{p'}\\
&\leq K\Big(\frac{R^{2m-1}}{R-r}T_\varphi(r,r_0)\Big)^{p'}.
\end{aligned}
$$

□

To complete the proof of Theorem 1.3.2, we need

Lemma 1.3.8 *Let φ be a nonzero meromorphic function on $B(R_0) \subseteq \mathbb{C}^m$ and $0 < r_0 <$*

$r < R < R_0$. *Then*

$$T_{D_i\varphi}(r, r_0) \le T_\varphi(r, r_0) + K \log^+ \Big(\frac{R^{2m-1}}{R-r} T_\varphi(R, r_0) \Big) \qquad for \quad i = 1, 2, ..., m.$$

Proof: Using the fact that for two meromorphic functions φ_1, φ_2 on $B(R_0)$, $T_{\varphi_1\varphi_2}(r, r_0) \le T_{\varphi_1}(r, r_0) + T_{\varphi_2}(r, r_0) + K$ and $\Big| T_\varphi(r, r_0) - \Big(\int_{S(r)} \log^+ |\varphi| \sigma_n + N_\varphi^\infty(r, r_0) \Big) \Big| \le K$, we see that

$$\begin{aligned} T_{D_i\varphi}(r, r_0) &\le T_{D_i\varphi/\varphi}(r, r_0) + T_\varphi(r, r_0) + K \\ &\le \int_{S(r)} \log^+ |D_i\varphi/\varphi| \sigma_m + N^\infty_{D_i\varphi/\varphi}(r, r_0) + T_\varphi(r, r_0) + K. \end{aligned}$$

On the other hand, since $N_\varphi^a(r, r_0) \le T_\varphi(r, r_0) + K$, we have

$$\begin{aligned} N^\infty_{D_i\varphi/\varphi}(r, r_0) &\le T_{\frac{D_i\varphi}{\varphi}}(r, r_0) \\ &= \int_{S(r)} \log \| \frac{D_i\varphi}{\varphi} \| \sigma_m + K. \end{aligned}$$

Since we have proved the Theorem for the case $|\alpha| = 1$, we use Corollary 1.3.3 in this case to get

$$\int_{S(r)} \log^+ |D_i\varphi/\varphi| \sigma_m \le K \log^+ \Big(\frac{R^{2m-1}}{R-r} T_\varphi(R, r_0) \Big).$$

We therefore conclude the Lemma. $\square$

Proof of Theorem 1.2.3 for the general case.

Proof: Assume that the Theorem holds for the case $|\alpha| \le k$. Take an arbitrarily α with $|\alpha| = k+1$ and write $D^\alpha = D^{\alpha'} D_i$, where $1 \le i \le m$ and $|\alpha'| = k$. Then $D^\alpha\varphi/\varphi = (D_i\varphi/\varphi)(D^{\alpha'}(D_i\varphi)/D_i\varphi)$, $\qquad z^\alpha = z_i z^{\alpha'}$ and $|\alpha|p = (|\alpha'|+1)p < p' < 1$. Set

$p_1 := 1/(|\alpha'|+1)$ and $p_2 := |\alpha'|/(|\alpha'|+1)$. By the Hölder inequality and the induction hypothesis, we have

$$\begin{aligned}
&\int_{S(r)} |z^{\alpha}(D^{\alpha}\varphi \diagup \varphi)(z)|^p \sigma_m(z) \\
&\leq \int_{S(r)} |z_i z^{\alpha'}(D_i\varphi/\varphi))(D^{\alpha'}(D_i\varphi)/D_i\varphi)|^p \sigma_m(z) \\
&\leq \int_{S(r)} |z_i D_i\varphi \diagup \varphi|^p |z^{\alpha'}(D^{\alpha'}(D_i\varphi)\diagup D_i\varphi)(z)|^p \sigma_m(z) \\
&= \int_{S(r)} \Big(|z_i D_i\varphi \diagup \varphi|^{p/p_1}\Big)^{p_1} \Big(|z^{\alpha'}(D^{\alpha'}(D_i\varphi)\diagup D_i\varphi)(z)|^{p/p_2}\Big)^{p_2} \sigma_m(z) \\
&\leq \int_{S(r)} \Big(|z_i D_i\varphi \diagup \varphi|^{p/p_1}\Big)^{p_1} \sigma_m \int_{S(r)} \Big(|z^{\alpha'}(D^{\alpha'}(D_i\varphi)\diagup D_i\varphi)(z)|^{p/p_2}\Big)^{p_2} \sigma_m \\
&\leq K\Big(\frac{R^{2m-1}}{R-r} T_{\varphi}(R,r_0)\Big)^{p'} \Big(\frac{R^{2m-1}}{R-r} T_{D_i\varphi}(R,r_0)\Big)^{p'}.
\end{aligned}$$

By Lemma 1.3.8,

$$T_{D_i\varphi}(R,r_0) \leq T_{\varphi} + K\log^+\Big(\frac{R^{2m-1}}{R-r} T_{\varphi}(R,r_0)\Big). \tag{1.4}$$

For $\epsilon > 0$ there exists a positive constant K_ϵ such that

$$\log^+\Big(\frac{R^{2m-1}}{R-r} T_{\varphi}(R,r_0)\Big) \leq K_\epsilon \Big(\frac{R^{2m-1}}{R-r} T_{\varphi}(R,r_0)\Big)^{\epsilon}.$$

We can conclude that

$$\int_{S(r)} |z^{\alpha}(D^{\alpha}\varphi \diagup \varphi)(z)|^p \sigma_m(z) \leq K\Big(\frac{R^{2m-1}}{R-r} T_{\varphi}(R,r_0)\Big)^{p'}$$

by the help of Lemma 1.3.8. This completes the proof of Theorem 1.3.2. $\square$

Definition 1.3.9 *Assume that f in nondegenerate. We say that $\{\alpha_1, ..., \alpha_{n+1}\}$ is an admissible set if $\{D^{\alpha_1} f, ..., D^{\alpha_{n+1}} f\}$ is a linearly independent set.*

Definition 1.3.10 *A meromorphic map $f : B(R_0) \subseteq \mathbb{C}^m \longrightarrow \mathbb{P}^n(\mathbb{C})$ is said to be (linearly) nondegenerate if $f(B_0) \nsubseteq H$ for every hyperplane H in $\mathbb{P}^n(\mathbb{C})$.*

Proposition 1.3.11 *(See [9], Proposition 4.5) Let $f : B(R_0) \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate meromorphic map. Then there exist $\alpha^j = (\alpha_{j1}, \dots, \alpha_{jm})$ with $\alpha_{ji} \geq 0$ being integers, and $|\alpha^1| + \cdots + |\alpha^{n+1}| \leq n(n+1)/2$ such that the generalized Wronskian $W_{\alpha^1 \cdots \alpha^{n+1}}(f) \not\equiv 0$.*

Lemma 1.3.12 *(See [8], Lemma 3.3) Let $f_0, ..., f_n$ be non-zero holomorphic functions on the unit disc $B(1)$ in $\mathbb{C}^m$, and set $\varphi_i = \dfrac{f_i}{f_n}(1 \leq i \leq n)$. Then, there is a polynomial $P(..., u_{il}, ...)$ with positive real coefficients not depending on each $f_0, ..., f_n$ such that*

$$\Big|\frac{W(f_0, ..., f_n)}{f_0 ... f_n}\Big| \leq P\Big(..., \Big|\Big(\frac{\varphi_i'}{\varphi_i}\Big)^{(l-1)}\Big|, ...\Big).$$

Definition 1.3.13 *A holomorphic map $f : B(1) \longrightarrow \mathbb{P}^n(\mathbb{C})$ is called transcendental if*

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r)}{\log(1/(1-r))} = \infty.$$

Proposition 1.3.14 *(See [8], Proposition 2.5) Let φ be a nowhere zero holomorphic function on the unit disk $\triangle(1) \subset \mathbb{C}$ which is not transcendental. Then, for each positive integer l, there exist a positive constant K_0 such that*

$$\int_0^{2\pi} \Big|\frac{d^{l-1}}{dz^{l-1}}\Big(\frac{\varphi'}{\varphi}\Big)(re^{i\theta})\Big| d\theta \leq \frac{K_0}{(1-r)^l} \log \frac{1}{1-r} \qquad (0 < r < 1).$$

Lemma 1.3.15 *(See [8], Lemma 3.4) Let $\varphi_1, \varphi_2, ..., \varphi_k$ be nowhere zero holomorphic functions on the unit disc $\triangle(1)$, $l_1, ..., l_k$ be positive real number with $kt < 1$. Assume that $\varphi_1, ..., \varphi_k$ are not transcendental. Then there exist a positive constant K_3 such that*

$$\int_0^{2\pi} \Big|\Big(\Big(\frac{\varphi_1'}{\varphi}\Big)^{(l_1-1)} ... \Big(\frac{\varphi_k'}{\varphi_k}\Big)^{(l_k-1)}\Big)(re^{i\theta})\Big|^t d\theta \leq \frac{K_3}{(1-r)^s}\Big(\log \frac{1}{1-r}\Big)^s,$$

$0 < r < 1$ and $s = t(l_1 + ... + l_k)$.

Take $\alpha^j = (\alpha_{j1}, \ldots, \alpha_{jm}), 1 \leq j \leq n+1$, so that the generalized Wronskian $W_{\alpha^1 \cdots \alpha^{n+1}}(f) \not\equiv 0$. Let $L_1, \ldots, L_q$ be linear forms of $n+1$-variables. Theorem 1.3.2 implies :

Proposition 1.3.16 *(See [9], Proposition 6.1) In the above situation, set $l_0 = |\alpha^1| + \cdots + |\alpha^{n+1}|$ and take t, p' with $0 < tl_0 < p' < 1$. Then, for $0 < r_0 < R_0$ there exists a positive constant K such that for $r_0 < r < R < R_0$,*

$$\int_{S(r)} \Big| z^{\alpha^1 + \cdots + \alpha^{n+1}} \frac{W_{\alpha^1 \ldots \alpha^{n+1}}(f)}{L_1(f) \cdots L_q(f)} \Big|^t \|f\|^{t(q-n-1)} \sigma_m \leq K \Big(\frac{R^{2m-1}}{R-r} T_f(R, r_0) \Big)^{p'}.$$

Definition 1.3.17 *Let $H_1, ..., H_q$ or $\mathbf{a_1}, ...\mathbf{a_q}$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ with coefficients vectors $\mathbf{a_1}, ..., \mathbf{a_q}$ in $\mathbb{C}^{n+1}$. We say that $H_1, ..., H_q$ are in general position if for any injective map $\mu : \{0, 1, ..., n\} \longrightarrow \{1, ..., q\}$, $\mathbf{a}_{\mu(0)}, ...\mathbf{a}_{\mu(n)}$ are linearly independent.*

Lemma 1.3.18 *(cf. [17]) Let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Then*

$$\sum_{j=1}^{q} N_f(r, H_j) - N_W(r, 0) \leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j),$$

where W denotes the Wronskian of f.

We now give the Second Main Theorem for meromorphic maps on the ball in $\mathbb{C}^m$ (cf. [9]).

Theorem 1.3.19 *Let $f : B(R_0) \longrightarrow \mathbb{P}^n(\mathbb{C})$ with $0 < R_0 \leq \infty$, be a meromorphic map which is non-degenerate and $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Then,*

$$(q - n - 1) T_f(r, r_0) \leq \sum_{j=1}^{q} N_f^{H_j}(r, r_0)^{[n]} + S_f(r),$$

where $S_f(r)$ is evaluated as follows.

(i) In the case $R_0 < \infty$,

$$S_f(r) \leq C\Big(\log^+ \frac{1}{R_0 - r} + \log^+ T_f(r, r_0)\Big)$$

for every $r \in [0, R_0)$ excluding a set E with $\int_E 1/(R_0 - t)dt < \infty$ and where C is a constant.

(ii) In the case $R_0 = \infty$,

$$S_f(r) \leq C\Big(\log^+ T_f(r, r_0) + \log r\Big)$$

for every $r \in [0, \infty)$ excluding a set E' with $\int_{E'} dt < \infty$ and where C is a constant.

Proof: Let

$$H_j : a_j^1 w_1 + ... + a_j^{n+1} w_{n+1} = 0 \text{ and } L_j(f) = a_j^1 f_1 + ... + a_j^{n+1} f_{n+1}$$

Using the concavity of the logarithm function, Proposition 1.3.16 implies that

$$\begin{aligned} t\int_{S(r)} \log |z^{\alpha^1+\cdots+\alpha^{n+1}}|\sigma_m + t\int_{S(r)} \log \Big|\frac{W_{\alpha^1 \ldots \alpha^{n+1}}(f)}{L_1(f)\cdots L_q(f)}\Big|\sigma_m \\ +t(q-n-1)\int_{S(r)} \log \|f\|\sigma_m \\ \leq \log \int_{S(r)} \Big|z^{\alpha^1+\cdots+\alpha^{n+1}} \frac{W_{\alpha^1 \ldots \alpha^{n+1}}(f)}{L_1(f)\cdots L_q(f)}\Big|^t \|f\|^{t(q-n-1)}\sigma_m \\ \leq C\Big(\log^+ \frac{R}{R-r} + \log^+ T_f(R, r_0)\Big). \end{aligned}$$

On the other hand, by Lemma 1.3.18, we have

$$-\sum_{j=1}^{q} N_f^{H_j}(r, r_0)^{[n]} \leq \int_{S(r)} \log \Big|\frac{W_{\alpha^1 \ldots \alpha^{n+1}}(f)}{L_1(f)\cdots L_q(f)}\Big|\sigma_m + C.$$

Therefore,

$$(q-n-1)T_f(r,r_0) \leq \sum_{j=1}^{q} N_f^{H_j}(r,r_0)^{[n]}+ \tag{1.5}$$

$$C\Big(\log r + \log^+ \frac{R}{R-r} + \log^+ T_f(R,r_0)\Big). \tag{1.6}$$

Since $T_f(r,r_0)$ is continuous, increasing function, we may assume that $T_f(r,r_0) \geq 1$.

If $R_0 < \infty$, we can apply Lemma 2.4 in [5] with $R = r + (R_0 - r)/eT_f(r,r_0)$ to get

$$T_f\Big(r + \frac{R_0 - r}{eT_f(r,r_0)}, r_0\Big) \leq 2T_f(r,r_0)$$

outside a set E containing r with $\int_E 1/(R_0 - t)dt < \infty$. Substituing $R = r + (R_0 - r)/eT_f(r,r_0)$ in (1.6) gives the desired inequality.

In the case $R_0 = \infty$, we apply Lemma 2.4 in [5] with $R = r + 1/T_f(r,r_0)$ to get

$$T_f\Big(r + \frac{1}{T_f(r,r_0)}, r_0\Big) \leq 2T_f(r,r_0)$$

outside a set E' containing r with $\int_{E'} dt < \infty$. Substituing $R = r + 1/T_f(r,r_0)$ in (1.6) gives the desired inequality. □

Definition 1.3.20 *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ with $f(\mathbb{C}) \nsubseteq H$ and m a positive integer or $+\infty$. We define the defect (truncated by m) of H for f by*

$$\delta_f(H)^{[m]} = 1 - \lim_{r\to\infty} \sup \frac{N_f(r,H)^{[m]}}{T_f(r)}.$$

For convenient's sake, we set $\delta_f(H)^{[m]} = 0$ if $f(\mathbb{C}) \subseteq H$ and, for brevity, we denote $\delta_f(H)^{[\infty]}$ by $\delta_f(H)$.

Note that we always have

$$0 \leq \delta_f(H)^{[m]} \leq 1$$

for every hyperplane H in $\mathbb{P}^n(\mathbb{C})$.

The following Corollary gives the defect relation.

Corollary 1.3.21 *In the same situation as in theorem 1.3.19, if*

(i) $R_0 < \infty$ *and*

$$\lim_{r \longrightarrow R_0} \sup \frac{T_f(r, r_0)}{\log 1/(R_0 - r)} = \infty$$

or (ii) $R_0 = \infty$, *then*

$$\sum_{j=1}^{q} \delta_f^{[n]}(H_j) \leq n + 1.$$

Proof: Theorem 1.3.19, implies that

$$\sum_{j=1}^{q} \Big(1 - \frac{N_f^{H_j}(r_0, r)^{[n]}}{T_f(r_0, r)}\Big) \leq n + 1 + \frac{S_f(r)}{T_f(r_0, r)}.$$

To conclude the proof, we observe that in the case (i), it is proven is [13], Proposition 5.5 that

$$\lim_{r \longrightarrow R_0} \sup \frac{T_f(r, r_0)}{\log(1/R_0 - r)} = \infty$$

is equivalent to

$$\lim_{r \longrightarrow R_0, r \notin E} \sup \frac{T_f(r, r_0)}{\log(1/(R_0 - r))} = \infty$$

for a set E with $\int_E 1/(R_0 - r)dr < \infty$.

In the case (ii), it always hold that

$$\lim_{r \longrightarrow \infty} \sup \frac{T_f(r, r_0)}{\log(r)} = \infty.$$

$\square$

In the following, we will give a generalization of Nevanlinna's five points uniqueness theorem to the case of holomorphic maps of $B(R_0) \subseteq \mathbb{C}^m$ into projective space sharing hyperplanes in general position.

Let consider hyperplanes

$$H_j := a_{j0}w_0 + ... + a_{jn}w_n \quad (1 \leq j \leq q)$$

in $\mathbb{P}^n(\mathbb{C})$ which are in general position and satisfy the condition

$$\dim g^{-1}(H_i \cap H_j) \leq m - 2.$$

Let f and g be a nondegenerate meromorphic map of $B(R_0) \subseteq \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ satisfying the conditions:

$$(i) \min(\nu(f, H_j), 1) = \min(\nu(g, H_j), 1) \ for\ 1 \leq j \leq q \ \ and \tag{1.7}$$

$$(ii) f = g \ \ on \ \ \cup_{j=1}^{q} g^{-1}(H_j). \tag{1.8}$$

With each $c = (c_0, ..., c_n) \in \mathbb{C}^{n+1} - \{0\}$ associate a hyperplane

$$H_c := \{[w_0 : ... : w_n]; \sum c_j w_j = 0\}$$

and define

$$\mathcal{C} = \{c \in \mathbb{C}^{n+1} - \{0\}; \dim f^{-1}(H_c \cap H_j) \leq m - 2 \ \text{ for } 1 \leq j \leq q\}.$$

Lemma 1.3.22 *(cf. [6]) The set $\mathcal{C}$ is dense in $\mathbb{C}^{n+1} - \{0\}$.*

We now prove the following uniqueness theorem:

Theorem 1.3.23 *(See [7], Corollary 4.8)*

If $q \geq 3n+2$, then $f = g$.

Proof: Assume that $f \neq g$. Take an arbitrary $c \in \mathcal{C}$ and define

$$F_{H_j} := \frac{a_j^0 f_0 + ... + a_j^n f_n}{c_0 f_0 + ... + c_n f_n}$$

and

$$G_{H_j} := \frac{a_j^0 g_0 + ... + a_j^n g_n}{c_0 g_0 + ... + c_n g_n}$$

By Proposition 1.3.19 applied to f and g we have:

$$(q-n-1)T_f(r) \leq \sum_{j=1}^{q} N_f(r, H_j)^{[n]} + S_f(r),$$

$$(q-n-1)T_g(r) \leq \sum_{j=1}^{q} N_g(r, H_j)^{[n]} + S_g(r).$$

Adding these two inequalities and using the above assumptions, we get that:

$$\begin{aligned}(q-n-1)(T_g(r)+T_f(r)) &\leq \sum_{j=1}^{q}(N_g(r,H_j)^{[n]} + N_f(r,H_j)^{[n]}) + S_g(r) + S_f(r)\\ &\leq n(N_{F_{H_{j0}}}(r,0)^{[1]} + N_{G_{H_{j0}}}(r,0)^{[1]}) + S_g(r) + S_f(r)\\ &\leq 2nN_{F_{H_{j0}}-G_{H_{j0}}}(r,0) + S_g(r) + S_f(r).\end{aligned}$$

In the second inequality, we have used the fact that a nondegenerate meromorphic map can not omit $3n$+2 hyperplanes in general position. By the first main theorem, $N_{F_{H_{j0}}-G_{H_{j0}}}(r,0) \leq T_{F_{H_{j0}}-G_{H_{j0}}}(r) + O(1) \leq T_f(r) + T_g(r) + O(1)$. So

$$(q-3n-1)(T_f(r)+T_g(r)) \leq S_g(r) + S_f(r) + O(1).$$

Divide both sides by $T_f(r)+T_g(r)$, we then obtain

$$q-3n-1 \leq \frac{S_g(r)+S_f(r)+O(1)}{T_f(r)+T_g(r)}. \tag{1.9}$$

In the case $R = \infty$, taking the limit as r approaches ∞ in (1.9) leads to a contradiction. In the case $R < \infty$ if in addition we assume $\lim_{r \longrightarrow R_0} \frac{T_f(r,r_0)}{\log(1/(R_0-r))} = \infty$, then again taking limit as r approaches R_0 in (1.9) leads to a contradiction.

$\square$

Chapter 2

Meromorphic map of complete k$\ddot{a}$hler manifolds into projective space

In this chapter, we'll give a non-integrated defect relation for a meromorphic map f on a complete Kähler manifold, whose universal covering is biholomorphic to the ball in $\mathbb{C}^m$, into $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces in general position. We first remark that in general Nevanlinna theory introduced in chapter 1 doesn't work on non-parabolic type complex manifolds since $\lim_{r\longrightarrow R} \frac{T_f(r)}{\log 1/(R-r)}$ may be finite . Example of this is seen in the study of the Guass map of complete minimal surfaces, where the order function is defined via the Ricci form on M. In order to develop a Nevanlinna theory for the above mentioned Kähler manifold, we assume the following growth condition for f: there exists a nonzero bounded continuous real-valued function h on M such that $\rho\Omega_f + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h^2 \geq \mathrm{Ric}\,\omega$ for some non-negative constant ρ, where Ω_f is the pull-back of the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$ and Ric ω is the Ricci form of the Kähler form ω.

2.1 Non-integrated defect for meromorphic maps

2.1.1 Currents and plurisubharmonic functions on complex manifolds

In this section we introduce the notion of currents on an $m-$dimensional complex manifold M. The point here is to establish a relationship between complex submanifolds (or subvarieties) and smooth differential forms. These two are connected by the notion of distributions or generalized function for the case $n = 1$ and currents for the case $n \geq 1$.

Let $f, g \in \mathcal{C}^0(\mathbb{R})$. From Calculus, we have

$$f \equiv g \Longleftrightarrow \int_{\mathbb{R}} f(x)\phi(x) = \int_{\mathbb{R}} g(x)\phi(x) \qquad \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}).$$

Let $A, B \subset \mathbb{R}$ be closed intervals. Then

$$A = B \Longleftrightarrow \int_A \phi(x)dx = \int_B \phi(x)dx \qquad \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}).$$

From these two observations, we can see that whether T is a function or an interval, it can be viewed as a linear functional: $T_1 = T_2 \Longleftrightarrow T_1(\phi) = T_2(\phi) \ \ \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R})$. Such a function ϕ is called a test function. This gives the notion of "generalized function" ($n = 1$) and "current" (for $n \geq 1$).

Let $C_c^\infty(\mathbb{R}^m)$ be the vector space of compactly supported smooth functions on $\mathbb{R}^m$. If $x = (x_1, \cdots, x_m)$ are coordinates on $\mathbb{R}^m$, we let $D_i = \partial/\partial x_i$ and $D^\alpha = D_1^{\alpha_1} \cdots D_1^{\alpha_m}$ for $\alpha = (\alpha_1, \cdots, \alpha_m) \in (\mathbb{Z}^+)^m$. The C^p-topology is defined on $C_c^\infty(\mathbb{R}^m)$ by saying that a sequence $\varphi_n \longrightarrow 0$ in case there is a compact set K with all supp$\varphi_n \subset K$ and with

$$D^\alpha \varphi_n(x) \longrightarrow 0$$

uniformly for $x \in K$ and all α satisfying $\alpha_1 + \cdots + \alpha_m \leq p$. The $C^\infty-$topology is defined by saying that $\varphi_n \longrightarrow 0$ in case all supp$\varphi_n \subset K$ and $\varphi_n \longrightarrow 0$ in the C^p-topology for each p.

Definition 2.1.1 *A distribution on $\mathbb{R}^m$ is a linear map $T : \mathcal{C}_c^\infty(\mathbb{R}^m) \longrightarrow \mathbb{C}$ that is continuous in the C^∞ topology. The distribution is said to be of order p if it is continuous in the C^p-topology.*

(p,q)-currents on complex manifolds

Let

$$\mathcal{E}^{p,q}(M) = \{smooth \ \ (p,q) - forms \ \ on \ \ \ M\}$$
$$\mathcal{D}^{p,q}(M) = \{smooth \ \ (p,q) - forms \ \ on \ \ M \ \ with \ \ \ compact \ \ \ support\}$$

and the dual space of $\mathcal{D}^{p,q}(M)$ is defined by:

$$(\mathcal{D}^{p,q}(M))^\star = \mathcal{D}'_{p,q}(M) = (\mathcal{D}')^{m-p,m-q}(M).$$

An element $\Theta \in \mathcal{D}'_{p,q}(M)$ is called a current of bidegree $(m-p, m-q)$ or bidimension (p,q) or simply a $(m-p, m-q)$-current. A (p,p) current T is real in case $T = \overline{T}$ in the sense that $\overline{T(\varphi)} = T(\overline{\varphi})$ for all $\varphi \in \mathcal{D}^{p,p}(M)$ and a real current is positive in the case

$$(\sqrt{-1})^{p(p-1)/2} T(\varphi \wedge \overline{\varphi}) \geq 0, \qquad \varphi \in \mathcal{D}^{m-p,0}(M).$$

Especially noteworthy are the closed, positive currents. Note that for the real current T of type (p,p),

$$dT = 0 \Longleftrightarrow \partial T = \overline{\partial} T = 0.$$

The positivity of a current implies that it is of order zero in the sense of distributions. For example, a current $T \in \mathcal{D}^{1,1}(M)$ is locally written as

$$T = \frac{\sqrt{-1}}{2} \sum_{i,j} t_{ij} dz_i \wedge d\overline{z}_j,$$

a differential form with distribution coefficients defined by

$$t_{ij}(\alpha) = (-1)^{n+i+j}(\alpha dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_m \wedge d\overline{z}_1 \wedge \cdots \widehat{d\overline{z}_j} \wedge \cdots d\overline{z}_m).$$

The current is real if $\overline{t_{ij}} = t_{ji}$, and positive if for any $\lambda_1, \cdots \lambda_m$, the distribution

$$\alpha \mapsto T(\lambda)(\alpha) = \Big(\sum_{i,j} t_{ij}\lambda_i\overline{\lambda}_j\Big)(\alpha)$$

is nonnegative on positive functions.

For $\Theta \in \mathcal{D}'^q(M)$, we define the exterior derivative to be the current $d\Theta \in \mathcal{D}'^{q+1}(M)$ given by

$$d\Theta(\varphi) := (-1)^{q+1}\Theta(d\varphi), \ \ \forall\ \varphi \in \mathcal{D}^{m-q-1}(M).$$

A real function $\varphi \in L^1(M, loc)$ is said to be plurisubharmonic in case $\sqrt{-1}\partial\overline{\partial}\varphi$ is a positive $(1,1)$ current.

2.1.2 Non-integrated defect for meromorphic maps

Let M be an m-dimensional complex Kăhler manifold. Let f be a meromorphic map of M into $\mathbb{P}^n(\mathbb{C})$, μ_0 be a positive integer and D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d with $f(M) \not\subset D$. We denote the intersection multiplicity of the image of f and D at $f(p)$ by $\nu^f(D)(p)$ and the pull-back of the normalized Fubini-Study metric form Ω on $\mathbb{P}^n(\mathbb{C})$ by Ω_f. The *non-integrated defect of f with respect to D cut by μ_0* is defined by

$$\delta^f_{\mu_0}(D) := 1 - \inf\{\eta \geq 0 : \eta \text{ satisfies condition } (\star)\}.$$

Here, the condition $(\star)$ means that there exists a bounded nonnegative continuous function h on M with zeros of order not less than $\min(\nu^f(D), \mu_0)$ such that

$$d\eta\Omega_f + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h^2 \geq [\min(\nu^f(D), \mu_0)],$$

where d is the degree of D and we mean by $[\nu]$ the $(1,1)$-current associated with a divisor ν. Note that the condition $(\star)$ also means that, for each holomorphic function $\phi(\not\equiv 0)$ on an open subset U of M with $\nu_\phi = \min(\nu^f(D), \mu_0)$ outside an analytic set of codimension ≥ 2, the function $u := \log(h^2\|f\|^{2d\eta}/|\phi|^2)$ is continuous and plurisubharmonic on U, where $\|f\|^2 = |f_0|^2 + \cdots + |f_n|^2$, and $f = [f_0 : \cdots : f_n]$ is a (local) reduced representation of f. So, similar to the classical Nevanlinna's defect, we have the following properties:

- $0 \leq \delta^f_{\mu_0}(D) \leq 1$. To see $\delta^f_{\mu_0}(D) \geq 0$, take $\eta = 1$ and $h = |Q(f)|/\|f\|^d$, where Q is the homogeneous polynomial defining D;
- If $f(M) \cap D = \emptyset$, then, by taking $\eta = 0, h = 1$, we have that $\delta^f_{\mu_0}(D) = 1$;
- If $\nu^f(D)(p) \geq \mu$ for all $p \in f^{-1}(D)$, with some positive integer $\mu \geq \mu_0$, then $\delta^f_{\mu_0}(D) \geq 1 - \mu_0/\mu$ by taking $\eta = \mu_0/\mu$ and $h = |Q(f)|/\|f\|^{\mu_0 d/\mu}$.

The relationship between the non-integrated defect and the classical Nevanlinna's defect is given as follows.

Proposition 2.1.2 *If* $\lim_{r\to R_0} T_f(r, r_0) = \infty$, *then*

$$0 \leq \delta^f_{\mu_0}(D) \leq \delta^{f,\star}_{\mu_0}(D) \leq 1,$$

where $\delta^\star$ *is classical Nevanlinna's defect.*

Proof: Take η satisfying the condition $(\star)$ in the definition of $\delta^f_{\mu_0}(D)$. The function

$$v := d\eta \log\|f\| + \log h - \log|\varphi|$$

is then plurisubharmonic, where h is bounded and φ is holomorphic on $B(R_0)$ with $\nu_\varphi = \min(\nu^f(D), \mu_0)$ outside an analytic set of codimension ≥ 2. Therefore,

$$\begin{aligned} 0 &\leq \int_{S(r)} v\sigma_m - \int_{S(r_0)} v\sigma_m \\ &= d\eta \int_{S(r)} \log\|f\|\sigma_m + \int_{S(r)} \log h\sigma_m - \int_{S(r)} \log|\varphi|\sigma_m + K \\ &\leq d\eta T_f(r, r_0) - N_f^{[\mu_0]}(r, D) + K \end{aligned}$$

where K is a constant, because h is bounded from above. This implies that

$$\frac{N_f^{[\mu_0]}(r, D)}{dT_f(r, r_0)} \leq \eta + \frac{K}{T_f(r, r_0)}.$$

As $r \to R_0$, we obtain $\delta^\star_{\mu_0}(D) \geq 1 - \eta$. Hence $\delta^\star_{\mu_0}(D) \geq \delta^f_{\mu_0}(D)$. $\square$

Let $D_1, ..., D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d, located in general position, and let $Q_j, 1 \leq j \leq q$, be homogeneous polynomials defining D_j. Let N be a large integer (to be determined later), and let V_N be the space of homogeneous polynomials of $n+1$ variables of degree N. Pick n distinct polynomials $\gamma_1, ..., \gamma_n \in \{Q_1, .., Q_q\}$. Arrange the $n-$tuples $\mathbf{i} = (i_1, ..., i_n)$ of non-negative integers by lexicographic order. Define, for the $n-$tuples $\mathbf{i} = (i_1, ..., i_n)$ of non-negative integers with $\sigma(\mathbf{i}) := \sum_j i_j \leq N/d$, the spaces $W_{\mathbf{i}} := W_{N,\mathbf{i}}$ by

$$W_{N,\mathbf{i}} = \sum_{\mathbf{e} \geq \mathbf{i}} \gamma_1^{e_1} \cdots \gamma_n^{e_n} V_{N - d\sigma(\mathbf{e})}.$$

Clearly, $W_{(0,...,0)} = V_N$ and $W_{\mathbf{i}} \supset$ '$W_{(\mathbf{i}')}$ if $\mathbf{i}' \geq \mathbf{i}$, so that the $\{W_{\mathbf{i}}\}$ in fact defines a filtration of V_N. We recall the following lemma due to [14].

Lemma 2.1.3 *(See [14], Proposition 3.3) For any nonnegative integer N and any $\{\gamma_1, ..., \gamma_n\} \subset \{Q_1, .., Q_q\}$, the dimension of the vector space*

$$\frac{V_N}{(\gamma_1, ..., \gamma_n) \cap V_N}$$

is equal to the number of $n-$*tuples* $(\mathbf{i}) = (i_1, ..., i_n) \in \mathbb{Z}^n_{\geq 0}$ *such that* $i_1 + \cdots + i_n \leq N$ *and* $0 \leq i_1, ..., i_n \leq d-1$. *In particular, for all* $N \geq n(d-1)$, *we have*

$$\dim \frac{V_N}{(\gamma_1, ..., \gamma_n) \cap V_N} = d^n.$$

Lemma 2.1.4 *(See [16], Lemma 3.2) There is an isomorphism*

$$\frac{W_{\mathbf{i}}}{W_{\mathbf{i}'}} \cong \frac{V_{N-d\sigma(\mathbf{i})}}{(\gamma_1, ..., \gamma_n) \cap V_{N-d\sigma(\mathbf{i})}},$$

where $\mathbf{i}' > \mathbf{i}$ *are consecutive* $n-$*tuples with* $W'_{\mathbf{i}} \subset W_{\mathbf{i}}$.

Let $\triangle_{\mathbf{i}} = \dim(W_{\mathbf{i}}/W_{\mathbf{i}'})$, where $\mathbf{i}' > \mathbf{i}$ are consecutive $n-$tuples with $W_{\mathbf{i}'} \subset W_{\mathbf{i}}$. By lemma 2.1.3, $\triangle_{\mathbf{i}} = d^n$ for every $\mathbf{i}$ such that $N - d\sigma(\mathbf{i}) \geq n(d-1)$. Moreover, Lemma 4.1 implies that $\triangle_{\mathbf{i}}$ is independent of the choice of $\gamma_1, ..., \gamma_n$. Hence, $\sum_{\mathbf{i}} i_j \triangle_{\mathbf{i}}$ is independent of the choice of $\gamma_1, ..., \gamma_n$ and j for $j = 1, \ldots, n$. Set, for $1 \leq j \leq n$,

$$\triangle := \sum_{\mathbf{i}} i_j \, \triangle_{\mathbf{i}} \,. \tag{2.1}$$

Lemma 2.1.5 *With* $N = 2d(n+1)(nd+n)(2^n - 1)(I(\epsilon^{-1}) + 1) + nd$ *for any* $\epsilon > 0$, *we have*

$$\frac{lN}{\triangle} \leq d(n+1) + \epsilon/2, \tag{2.2}$$

where $l = \begin{pmatrix} N+n \\ n \end{pmatrix}$. *Moreover,* l *satisfies the following estimate*

$$l \leq 2^{n^2+4n} e^n d^{2n} (nI(\epsilon^{-1}))^n, \tag{2.3}$$

where $I(x) := \min\{k \in \mathbb{N} : k > x\}$ *for a positive real number* x.

Proof: First notice that

$$l = \begin{pmatrix} N+n \\ n \end{pmatrix} = \frac{(n+N)(n+N-1)\cdots(n+1)N!}{N!n!} \leq \frac{(N+n)^n}{n!}. \tag{2.4}$$

Now since N is divisible by d, it follows from lemma 2.1.3 that,

$$\begin{aligned}
\triangle = \sum_{\sigma(\mathbf{i})\leq N/d} i_j \triangle_{\mathbf{i}} \geq \sum_{\sigma(\mathbf{i})\leq N/d-n} i_j \triangle_{\mathbf{i}} = d^n \sum_{\sigma(\mathbf{i})\leq N/d-n} i_j \\
= \frac{d^n}{n+1} \sum_{\sigma(\hat{\mathbf{i}})=N/d-n} \sum_{j=1}^{n+1} i_j \\
= \frac{d^n}{n+1} \sum_{\sigma(\hat{\mathbf{i}})=N/d-n} (N/d-n) \\
= \frac{d^n}{n+1} \begin{pmatrix} N/d \\ n \end{pmatrix} (N/d-n) \\
= \frac{N(N-d)\cdots(N-dn)}{d(n+1)!},
\end{aligned}$$

where $\hat{\mathbf{i}} = (i_1, \ldots, i_{n+1})$ and, in above, we used the fact that the number of nonnegative integer m-tuples with sum $\leq T$ for a positive integer T is equal to the number of non-negative integer $(m+1)$-tuples with sum exactly T, which is $\begin{pmatrix} T+m \\ m \end{pmatrix}$.

For every integer $j \leq n$, $(N-dj) \geq (N-dn)$; so

$$\prod_{j=1}^{n} \frac{1}{N-dj} \leq \Big(\frac{1}{N-dn}\Big)^n$$

and thus

$$\frac{lN}{\triangle} \leq d(n+1)\Big(\frac{N+n}{N-nd}\Big)^n.$$

Using

$$N = 2d(n+1)(nd+n)(2^n-1)(I(\epsilon^{-1})+1) + nd$$

one finds that

$$\begin{aligned}
\Big(\frac{N+n}{N-nd}\Big)^n &= \Big(1+\frac{n+nd}{N-nd}\Big)^n \\
&= 1+\sum_{r=1}^{n}\Big(\begin{array}{c} n \\ \\ r \end{array}\Big)\Big(\frac{nd+n}{N-nd}\Big)^r \\
&\le 1+(2^n-1)\frac{nd+n}{N-nd} \\
&\le 1+\frac{\epsilon}{2d(n+1)}.
\end{aligned}$$

Therefore

$$\frac{lN}{\triangle} \le d(n+1)+\epsilon/2.$$

To estimate l, we use following inequality

$$\left(\begin{array}{c} x+y \\ \\ y \end{array}\right) \le \frac{(x+y)^{x+y}}{x^x y^y} = \Big(1+\frac{y}{x}\Big)^x\Big(1+\frac{x}{y}\Big)^y = \Big(e(1+\frac{x}{y})\Big)^y$$

for positive integers x, y. Hence, with $N = 2d(n+1)(nd+n)(2^n-1)(I(\epsilon^{-1})+1)+nd$, we have

$$\begin{aligned}
l = \left(\begin{array}{c} N+n \\ \\ n \end{array}\right) &\le e^n\Big(1+\frac{N}{n}\Big)^n \\
&\le e^n\big(1+2d(n+1)(d+1)(2^n-1)(I(\epsilon^{-1})+1)+d\big)^n \\
&\le 2^{n^2+4n}e^n d^{2n}(nI(\epsilon^{-1}))^n.
\end{aligned}$$

□

The main result of this chapter is the following theorem:

Theorem 2.1.6 *Let M be an m-dimensional complete Kähler manifold and $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is algebraically nondegenerate (i.e. it's image is not* *contained in any proper subvariety of $\mathbb{P}^n(\mathbb{C})$). Assume that the universal covering of M is biholomorphic to a ball in $\mathbb{C}^m$. Let $D_1, ..., D_q$ be hypersurfaces of degree d_j in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $d = l.c.m.\{d_1, \ldots, d_q\}$ (the least common multiple of $\{d_1, \ldots, d_q\}$). Assume that, there exists a nonzero bounded continuous real-valued function h on M such that $\rho\Omega_f + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h^2 \geq \operatorname{Ric}\omega$ for some non-negative constant ρ. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \delta_{l-1}^{f}(D_j) \leq n + 1 + \epsilon + \frac{\rho l(l-1)}{d},$$

where $l \leq 2^{n^2+4n}e^n d^{2n}(nI(\epsilon^{-1}))^n$ and $I(x) := \min\{k \in \mathbb{N} : k > x\}$ for a positive real number x.

Proof: Since the universal covering of M is the unit ball in $\mathbb{C}^m$, by lifting f to the covering, we may assume that $M = B(1) \subset \mathbb{C}^m$. So we let $f : B(1) \longrightarrow \mathbb{P}^n(\mathbb{C})$ be an algebraically nondegenerate map. The proof of the main theorem breaks into the following two cases: the case

$$\lim_{r\longrightarrow 1}\sup \frac{T_f(r, r_0)}{\log 1/(1-r)} < \infty$$

and the case

$$\lim_{r\longrightarrow 1}\sup \frac{T_f(r, r_0)}{\log 1/(1-r)} = \infty.$$

We first deal with the case when

$$\lim_{r\longrightarrow 1}\sup \frac{T_f(r, r_0)}{\log 1/(1-r)} < \infty.$$

Let $D_1, ..., D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree $d_1, \ldots, d_q$, located in general position. Let $Q_j, 1 \leq j \leq q$, be the homogeneous polynomials defining D_j. Replacing Q_j by Q_j^{d/d_j} if necessary, where d is the l.c.m (the least common multiple) of d_j's, we can assume that $Q_1, \ldots, Q_q$ have the same degree d. For $N \in \mathbb{N}$, let V_N be the space of homogeneous polynomials of $n+1$ variables of degree N and fix a (arbitrary) basis $\phi_1, ..., \phi_l$, where $l = \dim V_N$. Set $F = [\phi_1(f) : \cdots : \phi_l(f)]$. Then $F : B(1) \to \mathbb{P}^{l-1}(\mathbb{C})$ is linearly non-degenerate. By Proposition 1.3.11, there exist $\alpha^j = (\alpha_{j1}, \ldots, \alpha_{jl})$ with $\alpha_{ji} \geq 0$ being integers, $|\alpha^j| \leq l-1$ for $1 \leq j \leq l$, and $|\alpha^1| + \cdots + |\alpha^l| \leq l(l-1)/2$ such that the generalized Wronskian $W_{\alpha^1 \cdots \alpha^l}(F) \not\equiv 0$.

Given $z \in B(1)$ there exists a numbering $\{i_1, ..., i_q\}$ of the indices $1, ..., q$ such that

$$|Q_{i_1} \circ f(z)| \leq \cdots \leq |Q_{i_q} \circ f(z)|. \tag{2.5}$$

Since $Q_1, ..., Q_q$ are in general position, Hilbert Nullstellensatz implies that for any integer $k, 0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} b_{jk}(x_0, ..., x_n) Q_{i_j}(x_0, ..., x_n),$$

where $b_{jk}, 1 \leq j \leq n+1, 0 \leq k \leq n$, are homogeneous forms with coefficients in $\mathbb{C}$ of degree $m_k - d$. So

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k - d} \max\{|Q_{i_1}(f)(z)|, ..., |Q_{i_{n+1}}(f)(z)|\},$$

where c_1 is a positive constant depending only on the coefficients of b_{jk}, thus depends only on the coefficients of Q_j. Therefore,

$$\|f(z)\|^d \leq c_1 \max\{|Q_{i_1}(f)(z)|, ..., |Q_{i_{n+1}}(f)(z)|\}. \tag{2.6}$$

By (2.5) and (2.6), we get

$$\prod_{j=1}^{q} \frac{\|f(z)\|^d}{|Q_j(f)(z)|} \leq c_1^{q-n} \prod_{k=1}^{n} \frac{\|f(z)\|^d}{|Q_{i_k}(f)(z)|}. \tag{2.7}$$

Take $\gamma_1 = Q_{i_1}, \cdots, \gamma_n = Q_{i_n}$ and let $V_N = W_{\mathbf{0}} \supset \cdots W_{\mathbf{i}} \supset W_{\mathbf{i}'} \supset \cdots$ be the filtration of V_N, associated to $\{\gamma_1, \ldots, \gamma_n\}$ as discussed earlier. We now choose a basis $\psi_1, ..., \psi_l$ for V_N in the following way: We start with the last nonzero $W_{\mathbf{i}}$ and pick a basis of it;

Then, we continue inductively as follows: suppose $\mathbf{i}' > \mathbf{i}$ are consecutive $n-$tuples such that $d\sigma(\mathbf{i}), d\sigma(\mathbf{i}') \leq N$ and assume that we have chosen a basis of $W_{(\mathbf{i}')}$; It follows directly from the definition that we may pick representatives in $W_{\mathbf{i}}$ for the quotient space $W_{\mathbf{i}}/W_{\mathbf{i}'}$, of the form $\gamma_1^{i_1}\cdots\gamma_n^{i_n}\eta$, where $\eta \in V_{N-d\sigma(\mathbf{i})}$. We extend the previously constructed basis in $W_{\mathbf{i}'}$ by adding these representatives. In particular we have obtained a basis for $W_{\mathbf{i}}$ and our induction procedure may go on unless $W_{\mathbf{i}} = V_N$. Note that if we let ψ be an element of the basis constructed with respect to $W_{\mathbf{i}}/W_{\mathbf{i}'}$, then we may write $\psi = \gamma_1^{i_1}\cdots\gamma_n^{i_n}\eta$, where $\eta \in V_{N-d\sigma(\mathbf{i})}$. Thus we have a bound

$$|\psi(f)(z)| \leq c_2|\gamma_1(f)(z)|^{i_1}\cdots|\gamma_n(f)(z)|^{i_n}\|f(z)\|^{N-d\sigma(i)} \tag{2.8}$$

where c_2 is a positive constant which depends only on f and $Q_1,\ldots,Q_q$. Observe that there are precisely $\triangle_{\mathbf{i}}$ such functions ψ in our basis. Write $\psi_1,...,\psi_l$ as linear forms $L_1,...,L_l$ in $\phi_1,...,\phi_l$ so that $\psi_t(f) = L_t(F)$, where $F = [\phi_1(f):\cdots:\phi_l(f)]$. Then (2.8) implies that

$$\prod_{t=1}^{l}\Big|L_t(F(z))\Big| \leq K\left(\prod_{\mathbf{i}=(i_1,\ldots,i_n)}|\gamma_1^{i_1}(f(z))\cdots\gamma_n^{i_n}(f(z))|^{\triangle_{\mathbf{i}}}\right)\|f(z)\|^{lN-d\sum_{\mathbf{i}}\sigma(\mathbf{i})\triangle_{\mathbf{i}}},$$

where, as we noted earlier, K is a constant depending only on f and $D_1,\ldots,D_q$ which may be different each time. So

$$\frac{\|f(z)\|^{d\sum_{\mathbf{i}}\sigma(\mathbf{i})\triangle_{\mathbf{i}}}}{\prod_{\mathbf{i}}|\gamma_1^{i_1\triangle_{\mathbf{i}}}(f(z))|\cdots|\gamma_n^{i_n\triangle_{\mathbf{i}}}(f(z))|} \leq K\frac{\|f(z)\|^{lN}}{\prod_{t=1}^{l}\Big|L_t(F)\Big|},$$

thus, using (4.1),

$$\frac{\|f(z)\|^{dn\triangle}}{|\gamma_1^{\triangle}(f(z))|\cdots|\gamma_n^{\triangle}(f(z))|} \leq K\frac{\|f(z)\|^{lN}}{\prod_{t=1}^{l}\Big|L_t(F(z))\Big|}.$$

With $\gamma_1 = Q_{i_1},\cdots,\gamma_n = Q_{i_n}$, it gives

$$\frac{\|f(z)\|^{dn\triangle}}{|Q_{i_1}^{\triangle}(f(z))\cdots Q_{i_n}^{\triangle}(f(z))|} \leq K\frac{\|f(z)\|^{lN}}{\prod_{t=1}^{l}\Big|L_t(F(z))\Big|}. \tag{2.9}$$

On the other hand, from (2.7),we get

$$\frac{\|f(z)\|^{dq\triangle}}{|Q_1^{\triangle}(f(z))\cdots Q_q^{\triangle}(f(z))|}\le K\frac{\|f(z)\|^{dn\triangle}}{|Q_{i_k}^{\triangle}(f)(z)\cdots Q_{i_n}^{\triangle}(f)(z)|}. \tag{2.10}$$

Combining (2.9) and (2.10), we derive that

$$\frac{\|f(z)\|^{dq\triangle}}{|Q_1^{\triangle}(f(z))\cdots Q_q^{\triangle}(f(z))|}\le K\frac{\|f(z)\|^{lN}}{\prod_{t=1}^{l}\Big|L_t(F(z))\Big|}.$$

Hence,

$$\frac{\|f(z)\|^{dq\triangle-lN}|W_{\alpha^1\ldots\alpha^l}(F)(z)|}{|Q_1^{\triangle}(f(z))\cdots Q_q^{\triangle}(f(z))|}\le K\frac{|W_{\alpha^1\ldots\alpha^l}(F)(z)|}{|L_1(F(z))\cdots L_l(F(z))|}. \tag{2.11}$$

Note that although $L_1,\ldots,L_l$ depend on z, there are only finitely many such choices since there are only finite choices of $\{\gamma_1,\ldots\gamma_n\}\subset\{Q_1,\ldots,Q_q\}$.

We continue with the proof of the Main Theorem by absurdity. We assume that

$$\rho\Omega_f+\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log h^2\ge \mathrm{Ric}\ \omega, \tag{2.12}$$

and

$$\sum_{j=1}^{q}\delta_{l-1}^{f}(D_j)>(n+1)+\epsilon+\frac{\rho l(l-1)}{d}. \tag{2.13}$$

Then, from the discussion earlier, there exist constants $\eta_j\ge 0$ and continuous plurisubharmonic functions $\tilde u_j(\not\equiv-\infty)$ such that $e^{\tilde u_j}|\varphi_j|\le\|f\|^{d\eta_j}$ for $j=1,...,q$, and

$$q-\sum_{j=1}^{q}\eta_j>n+1+\epsilon+\frac{\rho l(l-1)}{d}, \tag{2.14}$$

where φ_j is a nonzero holomorphic function with $\nu^0_{\varphi_j}=\min(\nu^f(D_j),l-1)$. Let $u_j=\tilde u_j+\log|\varphi_j|$. Then, $u_j(\not\equiv-\infty),1\le j\le q$, are continuous plurisubharmonic functions,

$$e^{u_j}\le\|f\|^{d\eta_j}, \tag{2.15}$$

and $u_j - \log|\varphi_j|$ is plurisubharmonic, where φ_j is a nonzero holomorphic function with $\nu^0_{\varphi_j} = \min(\nu^f(D_j), l-1)$. Let

$$v := \log\left| z^{\alpha^1+\cdots+\alpha^l} \frac{W_{\alpha^1\ldots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}\right| + \sum_{j=1}^{q} \triangle u_j, \tag{2.16}$$

where $\triangle$ is the integer defined in (4.1). We now show that v is plurisubharmonic on $M = B(1)$. To do so, we need the following lemma.

Proposition 2.1.7 *In the above situation, set*

$$\psi = \frac{W_{\alpha^1\cdots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}.$$

Then

$$\nu^{\infty}_{\psi} \leq \sum_{j=1}^{q} \triangle \min\{\nu^0_{Q_j(f)}, l-1\}$$

outside an analytic set of codimension at least two.

Proof: Let I_F be the indeterminacy set of F, and take $a \in B(1)\backslash I_F$. We first show the following claim: *For h a holomorphic function around a, assume that $D^{\alpha}h \not\equiv 0$ around a. Then* $\nu^0_{D^{\alpha}h}(a) = \max\{0, \nu^0_h(a) - |\alpha|\}$. To see this, take a system of holomorphic local coordinate $z = (z_1, \ldots, z_m)$ in a neighborhood of a such that $z(a) = 0$ and h can be written as $h = z_1^{\nu^0_h(a)}\tilde{h}$, and $\tilde{h}$ has no zero in a neighborhood of a. From this representation of h, we can easily conclude the claim.

Now for each $a \in B(1)\backslash I_F$, without loss of generality, we may assume that $Q_j(f)$ vanishes at a for $1 \leq j \leq q_1$ and $Q_j(f)$ does not vanish at a for $j > q_1$. By the assumption that the Q_j's are in general position, we know $q_1 \leq n$.

For $\{Q_1, ..., Q_n\} \subset \{Q_1, ..., Q_q\}$, consider the filtration $V_N = W_{\mathbf{0}} \supset \cdots W_{\mathbf{i}} \supset W_{\mathbf{i}'} \supset \cdots$, associated to $\{Q_1, \ldots, Q_n\}$ as discussed earlier, and take a basis $\psi_1, ..., \psi_l$ of V_N according to this filtration. Then, there are linearly independent linear forms $L_1, ..., L_l$ such that $\psi_t(f) = L_t(F), 1 \leq t \leq l$. Denote by $W := W_{\alpha^1 \dots \alpha^l}(F)$, the generalized Wronskian of F. From the basic properties of generalized Wronskian (see [9] Proposition 4.9),

$$W = W_{\alpha^1 \dots \alpha^l}(F) = CW_{\alpha^1 \dots \alpha^l}(L_1(F), \dots, L_l(F)) = CW_{\alpha^1 \dots \alpha^l}(\psi_1(f), ..., \psi_l(f)),$$

where C is some constant. Let ψ be an element of the basis $\{\psi_1, \dots, \psi_l\}$. As we discussed earlier, we may write $\psi = Q_1^{i_1} \cdots Q_n^{i_n} \eta$ with $\eta \in V_{N-d\sigma(i)}$. Therefore

$$\psi(f) = (Q_1(f))^{i_1} \cdots (Q_n(f))^{i_n} \eta(f),$$

and note that there are $\triangle_{\mathbf{i}}$ such ψ is our basis. Assume that $\nu^0_{Q_j(f)}(a) \geq l-1$ for $1 \leq j \leq q_0$ and $\nu^0_{Q_j(f)}(a) < l-1$ for $q_0 < j \leq q_1$. Since, from above, $W = C\det(D^{\alpha^i}(\psi_j(f)))_{1 \leq i,j \leq l}$, by the claim (note that there are $\triangle_{\mathbf{i}}$ such ψ is our basis), and noticing that $|\alpha^j| \leq l-1$ for $1 \leq j \leq l$,

$$\begin{aligned}
\nu^0_W(a) &\geq \sum_{\mathbf{i}} \left(\sum_{j=1}^{q_0} i_j(\nu^0_{Q_j}(a) - (l-1)) \right) \triangle_{\mathbf{i}} \\
&= \sum_{j=1}^{q_0} \left(\sum_{\mathbf{i}} i_j \triangle_{\mathbf{i}} \right) (\nu^0_{Q_j}(a) - (l-1)) = \triangle \sum_{j=1}^{q_0} (\nu^0_{Q_j}(a) - (l-1)).
\end{aligned}$$

On the other hand,

$$\sum_{j=1}^{q} \nu^0_{Q_j(f)}(a) = \sum_{j=1}^{n} \nu^0_{Q_j(f)}(a) = \sum_{j=1}^{q_0} \nu^0_{Q_j(f)}(a) + \sum_{j=q_0}^{q_1} \nu^0_{Q_j(f)}(a).$$

Hence, $\nu^\infty_\psi(a) \leq \sum_{j=0}^{q} \triangle \min\{\nu_{Q_j(f)}(a), l-1\}$. $\square$

From the above proposition, by the definition of v (see (2.16)), and using the fact that $u_j - \log|\varphi_j|$ is plurisubharmonic and $\nu^0_{\varphi_j} = \min(\nu^f(D_j), l-1)$, we see that v is plurisubharmonic on $M = B(1)$.

We now continue our proof. By the growth condition of f (see (2.12)), there exists a continuous plurisubharmonic function $w \not\equiv -\infty$ on $B(1)$ such that

$$e^w dV \leq \|f\|^{2\rho} v_m. \tag{2.17}$$

Set

$$t = \frac{2\rho}{qd \triangle - lN - \triangle d(\eta_1 + \cdots + \eta_q)}, \tag{2.18}$$

and

$$\chi := z^{\alpha^1 + \cdots + \alpha^l} \frac{W_{\alpha^1 \ldots \alpha^l}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)}.$$

Define

$$u := w + tv.$$

Then u is plurisubharmonic and so subharmonic on the Kähler manifold M.

By the result of S.T. Yau ([20]) and L. Karp ([22]), we have necessarily

$$\int_{B(1)} e^u dV = \infty,$$

because $B(1)$ has infinite volume with respect to the given complete Kähler metric (cf.[22], Theorem B). Now, from (2.15), (2.17) and (2.18)

$$\begin{aligned} e^u dV &= e^{w+tv} dV \leq e^{tv} \|f\|^{2\rho} v_m \\ &= |\chi|^t \Big(\prod_{j=1}^{q} e^{t\triangle u_j} \Big) \|f\|^{2\rho} v_m \leq |\chi|^t \Big(\prod_{j=1}^{q} \|f\|^{t\triangle d\eta_j} \Big) \|f\|^{2\rho} v_m \\ &= |\chi|^t \|f\|^{2\rho + td\triangle \sum_{j=1}^{q} \eta_j} v_m = |\chi|^t \|f\|^{t(dq\triangle - lN)} v_m. \end{aligned}$$

The contradiction will appear if we can show that

$$\int_{B(1)} e^u dV < \infty.$$

From the lemma 2.1.5, $\frac{lN}{\triangle} \leq d(n+1)+\epsilon$. Thus $qd - \frac{lN}{\triangle} \geq d(q-(n+1+\epsilon))$. So, using (2.14),

$$dq \triangle - lN - \triangle \sum_{j=1}^{q} d\eta_j \geq d \triangle (q-(n+1+\epsilon)) - \triangle \sum_{j=1}^{q} d\eta_j > \triangle \rho l(l-1),$$

This implies that $tl(l-1)/2 < 1$. Since $|\alpha^1| + \cdots + |\alpha^l| \leq l(l-1)/2$, we can choose p' such that $t(|\alpha^1| + \cdots + |\alpha^l|) \leq tl(l-1)/2 < p' < 1$. By the help of the identity (cf. [23], p.226),

$$v_m = (dd^c|z|^2)^m = 2m|z|^{2m-1}\sigma_m \wedge d|z|,$$

we have

$$\begin{aligned}
\int_{B(1)} e^u dV &\leq \int_{B(1)} |\chi|^t \|f\|^{t(dq\triangle - lN)} v_m \\
&\leq 2m \int_0^1 r^{2m-1} \Big(\int_{S(r)} |\chi|^t \|f\|^{t(dq\triangle - lN)} \sigma_m \Big) dr \\
&= 2m \int_0^1 r^{2m-1} \Big(\int_{S(r)} \Big| z^{\alpha^1 + \cdots + \alpha^l} \frac{W_{\alpha^1 \ldots \alpha^l}(F) \|f\|^{(dq\triangle - lN)}}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \sigma_m \Big|^t \Big) dr.
\end{aligned} \tag{2.19}$$

On the other hand, by (2.11),

$$\frac{|W_{\alpha^1 \ldots \alpha^l}(F)| \|f\|^{(dq\triangle - lN)}}{|Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)|} \leq K \sum_{L_1, \ldots, L_l} \left(\frac{|W_{\alpha^1 \ldots \alpha^l}(F)|}{|L_1(F) \cdots L_l(F)|} \right), \tag{2.20}$$

where the summation is taken for all the possible linear forms choices of the linear forms $L_1, \ldots, L_l$. Note that the set of linear forms $\{L_1, \ldots, L_l\}$ comes from the filtration of V_N associated to the $\{\gamma_1, \ldots, \gamma_n\} \subset \{Q_1, \ldots, Q_q\}$, hence the number of choices of the sets $\{L_1, \ldots, L_l\}$ is the same as the number of the choices of the sets $\{\gamma_1, \ldots, \gamma_n\}$, which is finite. Hence the summation in (2.20) is a finite sum whose number of terms depends only on f and $Q_1, \ldots, Q_q$. By Proposition 1.3.16, for each $L_1, \ldots, L_l$,

$$\int_{S(r)} \Big| z^{\alpha^1 + \cdots + \alpha^l} \frac{W_{\alpha^1 \ldots \alpha^l}(F)}{L_1(F) \cdots L_l(F)} \Big|^t \sigma_m \leq K \Big(\frac{R^{2m-1}}{R-r} T_F(R, r_0) \Big)^{p'}. \tag{2.21}$$

Combining (2.20) and (2.21) thus gives

$$\int_{S(r)} \Big| z^{\alpha^1+\cdots+\alpha^l} \frac{W_{\alpha^1\ldots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)} \Big|^t \|f\|^{t(dq\triangle - lN)} \sigma_m \leq K\Big(\frac{R^{2m-1}}{R-r} T_F(R, r_0)\Big)^{p'}, \tag{2.22}$$

for $r_0 < r < R < 1$, where, as we noted that, we use the letter K to denote a constant depending only on f and $D_1, \ldots, D_q$ even when it should be replaced by a new constant. According to Lemma 2.4 in [5], if we choose $R := r + (1-r)/eT_F(r, r_0)$, then

$$T_F(R, r_0) \leq 2T_F(r, r_0) \leq 2dT_f(r, r_0)$$

outside a set E with $\int_E 1/(1-r)dr < \infty$. If

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} < \infty,$$

then (2.22) becomes

$$\int_{S(r)} \Big| z^{\alpha^1+\cdots+\alpha^l} \frac{W_{\alpha^1\ldots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)} \Big|^t \|f\|^{t(dq\triangle - lN)} \sigma_m \leq \frac{K}{(1-r)^{p'}} \Big(\log \frac{1}{1-r}\Big)^{p'} \tag{2.23}$$

for all $r \in [0, 1)$ outside a set E with $\int_E 1/(1-r)dr < \infty$. Varying a constant K slightly, we may assume that the above inequality holds for all $r \in [0, 1)$ because of Proposition 5.5 in [9]. Therefore, by (2.19) and (2.23), we have

$$\int_{B(1)} e^u dV \leq K \int_0^1 \frac{r^{2m-1}}{(1-r)^{p'}} \Big(\log \frac{1}{1-r}\Big)^{p'} dr < \infty,$$

since $p' < 1$. This contradicts the result of S.T. Yau ([20]) and L. Karp ([22]) mentioned earlier. This completes the proof for the first case.

We now deal with the case where

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} = \infty.$$

This case is similar to the standard Nevanlinna theory. We use the logarithmic derivative lemma and the previous discussions to prove the following refinement of the Second Main Theorem (see [16]).

Theorem 2.1.8 *Let $f : B(R_0) \longrightarrow \mathbb{P}^n(\mathbb{C})$, $0 < R_0 \leq \infty$, be a meromorphic map which is algebraically nondegenerate and $D_1, ..., D_q$ be hypersurfaces of degree $d_j, 1 \leq j \leq q$, in $\mathbb{P}^n(\mathbb{C})$ located in general position. Then, for every $\epsilon > 0$,*

$$(q - (n + 1 + \epsilon))T_f(r, r_0) \leq \sum_{j=1}^{q} d_j^{-1} N_f^{[l-1]}(r, D_j) + S(r),$$

where $l \leq 2^{n^2+4n} e^n d^{2n} (nI(\epsilon^{-1}))^n$, $d = l.c.m\{d_1, ..., d_q\}$, and $S(r)$ is evaluated as follows:

(1) In the case $R_0 < \infty$,

$$S(r) \leq K\Big(\log^+ \frac{1}{R_0 - r} + \log^+ T_f(r, r_0)\Big)$$

for every $r \in [0, R_0)$ excluding a set E with $\int_E \frac{1}{R_0 - t} dt < \infty$.

(2) In the case $R_0 = \infty$,

$$S(r) \leq K(\log^+ T_f(r, r_0) + \log r)$$

for every $r \in [0, R_0)$excluding a set E' with $\int_{E'} dt < \infty$.

Proof: Without loss of generality, we assume that $d_1 = \cdots = d_q = d$. Similar to the proof of (2.22), by using (2.20) and Proposition 1.3.16, we have

$$\int_{S(r)} \Big| z^{\alpha^1 + \cdots + \alpha^l} \frac{W_{\alpha^1 \ldots \alpha^l}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \Big|^t \|f\|^{t(dq\triangle - lN)} \sigma_m \leq K\Big(\frac{R^{2m-1}}{R - r} T_F(R, r_0)\Big)^{p'}, \qquad (2.24)$$

for $r_0 < r < R < R_0$. Hence, by virtue of the concavity of the logarithm, the above inequality implies that

$$\begin{aligned} & t\int_{S(r)} \log |z^{\alpha^1 + \cdots + \alpha^l}| \sigma_m + t\int_{S(r)} \log \Big| \frac{W_{\alpha^1 \ldots \alpha^l}(F)}{Q_1^{\triangle}(f) \cdots Q_q^{\triangle}(f)} \Big| \sigma_m + \\ & t(dq \triangle - Nl) \int_{S(r)} \log \|f\| \sigma_m \\ & \leq K\Big(\log^+ \frac{R}{R - r} + \log^+ T_F(R, r_0)\Big) + O(1), \end{aligned} \qquad (2.25)$$

for $r_0 < R < R_0$. But, by the Jensen formula (see [9], p236),

$$\int_{S(r)} \log\Big|\frac{W_{\alpha^1\dots\alpha^l}(F)}{Q_1^{\triangle}(f)\cdots Q_q^{\triangle}(f)}\Big|\sigma_m = N_{\nu^0_{W_{\alpha^1\dots\alpha^l}(F)}}(r_0,r) - \triangle\sum_{j=1}^{q} N_f(r,D_j) + O(1).$$

By Proposition 2.1.7, $\triangle\sum_{j=1}^{q} N_f(r,D_j) - N_{\nu^0_{W_{\alpha^1\dots\alpha^l}(F)}}(r_0,r) \le \triangle\sum_{j=1}^{q} N_f^{[l-1]}(r,D_j)$ and therefore (2.25) becomes

$$(dq\triangle - Nl)T_f(r) \le \sum_{j=1}^{q}\triangle N_f^{[l-1]}(r,D_j) + K\Big(\log^+\frac{R}{R-r} + \log^+ T_F(R,r_0)\Big) + O(1).$$

By Lemma 2.1.5, with $N = 2d(n+1)(nd+n)(2^n-1)(I(\epsilon^{-1})+1)+nd$ for any $\epsilon > 0$, we have

$$\frac{lN}{\triangle} \le d(n+1)+\epsilon,$$

and moreover, l satisfies $l \le 2^{n^2+4n}e^n d^{2n}(nI(\epsilon^{-1}))^n$. Hence,

$$\begin{aligned}(q-(n+1+\epsilon))T_f(r) &\le \sum_{j=1}^{q} d^{-1}N_f^{[l-1]}(r,D_j) + K\Big(\log^+\frac{R}{R-r} + \log^+ T_F(R,r_0)\Big)\\ &\le \sum_{j=1}^{q} d^{-1}N_f^{[l-1]}(r,D_j) + K\Big(\log^+\frac{R}{R-r} + \log^+ T_f(R,r_0)\Big). \quad (2.26)\end{aligned}$$

Since $T_f(r,r_0)$ is continuous, increasing and we may assume $T_f(r,r_0) \ge 1$, we can apply Lemma 2.4 in [5] to show

$$T_f\Big(r + \frac{R_0-r}{eT_f(r,r_0)}, r_0\Big) \le 2T_f(r,r_0)$$

outside a set E of r such that $\int_E 1/(R_0-r)dr < \infty$ in the case $R_0 < \infty$ and

$$T_f\Big(r + \frac{1}{T_f(r,r_0)}, r_0\Big) \le 2T_f(r,r_0)$$

outside a set E' of r such that $\int_{E'} dr < \infty$ in the case $R_0 = \infty$. Substituting $R = r + \frac{R_0-r}{eT_f(r,r_0)}$ if $R_0 < \infty$ and $R = r + 1/T_f(r,r_0)$ if $R_0 = \infty$ in (2.26) proves the theorem. $\square$

Corollary 2.1.9 *In the same situation in Theorem 2.1.8, if*

$$(i) \quad \limsup_{r\to R_0} \frac{T_f(r, r_0)}{\log(1/R_0 - r)} = \infty$$

or

$$(ii) \quad R_0 = \infty,$$

then

$$\sum_j \delta^f_{l-1}(D_j) \leq \sum_j \delta^{f,\star}_{l-1}(D_j) \leq n + 1 + \epsilon,$$

where $\delta^{f,\star}$ is the classical Nevanlinna's defect defined by

$$\delta^{f,\star}_{l-1}(D_j) = \limsup_{r\to R_0} \left(1 - \frac{N_f^{[l-1]}(r, D_j)}{dT_f(r, r_0)}\right).$$

Corollary 2.1.9 gives the proof of the second case. The proof of the Main Theorem (Theorem 1.1) is thus complete. □

We remark that in the case $M = \mathbb{C}^m$ endowed with the flat metric, we also have the following statement (see Corollary 2.1.9) which is essentially due to Min Ru (see [16] and [18]) without the truncation and An-Phuong with the truncation (see [2]).

Theorem 2.1.10 *Let $f : \mathbb{C}^m \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is algebraically nondegenerate. Let $D_1, ..., D_q$ be hypersurfaces of degree d_j in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $d = l.c.m.\{d_1, \ldots, d_q\}$. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \delta^{f,\star}_{l-1}(D_j) \leq n + 1 + \epsilon,$$

where $l \leq 2^{n^2+4n} e^n d^{2n} (nI(\epsilon^{-1}))^n$, $I(x) := \min\{k \in \mathbb{N} : k > x\}$ for a positive real number x, and $\delta^{f,\star}_{l-1}(D)$ is the classical Nevanlinna's (truncated) defect of f with respect to D.

Note that, from the discussion above, we have that $\delta^{f}_{l-1}(D) \leq \delta^{f,*}_{l-1}(D)$ (see Proposition 2.1.2). Thus, Theorem 2.1.6 and Theorem 2.1.10 are complementing each other.

2.2 Uniqueness theorem

In 1926, R. Nevanlinna proved that for two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, they cannot share more than four distinct values; namely, in [1], the following unicity theorem for meromorphic functions on $\mathbb{C}$

Theorem 2.2.1 *Let ϕ, ψ be nonconstant meromorphic functions on $\mathbb{C}$. If there exist five distinct values $a_1, ..., a_5$ such that $\phi^{-1}(a_i) = \psi^{-1}(a_i)$ $(1 \leq i \leq 5)$, then $\phi \equiv \psi$*

Let M be a complete, connected Kähler manifold, whose universal covering is biholomorphic to the ball in $\mathbb{C}^m$. Generalization of the above theorem to the case of meromorphic maps of M into $\mathbb{P}^n(\mathbb{C})$ satisfying certain growth condition (see the condition (C_ρ) in Theorem 2.1.6) and sharing hyperplanes is given by Fujimoto (see [11]). He obtained a lower bound on the number of shared hyperplanes by two linearly nondegenerate meromorphic mappings on M to be identical. In this section, we extend the result in [11] to the case where the meromorphic maps share hypersurfaces instead of hyperplanes. The main result of this section is:

Theorem 2.2.2 *Let M be a complete, connected Kähler manifold whose universal covering is biholomorphic to the unit ball in $\mathbb{C}^m$, and let f and g be algebraically nondegenerate maps of M into $\mathbb{P}^n(\mathbb{C})$. If f and g satisfy the condition (C_ρ) and there exist q hypersurfaces*

$D_j, \quad j = 1, ..., q,$ of degree d located in general position in $\mathbb{P}^n(\mathbb{C})$ such that

$$(i) f = g \quad on \quad \cup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j)),$$
$$(ii) q > (n+1) + \frac{1}{2} + \frac{2n(l-1)}{d} + \frac{\rho l(l-1)}{d},$$

where $l \leq 2^{n^2+4n}(3n)e^n d^{2n}$, then $f \equiv g$.

Proof: For the proof of the Main Theorem, we may assume that $M = B(1)(\subset \mathbb{C}^m)$. Indeed, if $\pi : \tilde{M} \longrightarrow M$ is the universal covering map of M, then $\tilde{f} = f \circ \pi$ and $\tilde{g} = g \circ \pi$ also satisfy the assumption of the Main Theorem on the Kähler manifold $\tilde{M}$ and $\tilde{f} = \tilde{g}$ on $\tilde{M}$ implies $f = g$ on M. So we may assume that $M = \tilde{M}$. Let $f, g : B(1) \longrightarrow \mathbb{P}^n(\mathbb{C})$ be algebraically nondegenerate meromorphic maps satisfying all the assumptions in Theorem 2.2.2. We shall show that the assumption $f \neq g$ leads to a contradiction. The proof of the theorem breaks into the following two cases:

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} < \infty$$

and the case

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} = \infty.$$

We first deal with the case

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} < \infty.$$

Let

$$w = \sum_{i,j} h_{i\bar{j}} \frac{\sqrt(-1)}{2} dz_i \wedge d\bar{z}_j$$

be the given Kähler metric form on $B(1)$. By assumption there exist continuous plurisubharmonic functions u_1 and u_2 on $B(1)$ such that

$$e^{u_1}\det(h_{i\bar{j}})^{1/2} \leq \|f\|^{\rho}$$

$$e^{u_2}\det(h_{i\bar{j}})^{1/2} \leq \|g\|^{\rho}$$

Set $:= z^{\alpha^1+\ldots+\alpha^{n+1}}\Phi$ and $\tilde{\psi} := z^{\beta^1+\ldots+\beta^{n+1}}\Psi$, where

$$\Phi = \frac{W_\alpha(F)}{Q_1^{\triangle}(f)\ldots Q_q^{\triangle}(f)} \text{ and } \Psi = \frac{W_\beta(G)}{Q_1^{\triangle}(g)\ldots Q_q^{\triangle}(g)}.$$

Now, we choose distinct indices i_0 and j_0 such that

$$\chi := f_{i_0}g_{j_0} - f_{j_0}g_{i_0}. \tag{2.27}$$

Note that if $\chi \equiv 0$ for all indices i_0 and j_0, then $f \equiv g$. For χ not identically zero, if $\nu_{\Phi}^{\infty}(p) > 0$ for a point $p \in B(1)$, then $Q_j(f)(p) = 0$ for some $j \leq q$. Then $p \in f^{-1}(D_j) \subset \cup_{j=1}^{q}(f^{-1}(D_j)\cup g^{-1}(D_j))$ and so $f(p) = g(p) = 0$. This implies that $\chi(p) = 0$ and $\nu_{\Phi}^{\infty} \leq \nu_{\chi}^{0}$. Hence, by proposition 2.1.7 and the fact that the hypersurfaces are in general position, we can conclude that

$$N_{\Phi}^{\infty}(r, r_0) \leq \triangle n(l-1)N_{\chi}^{0}(r, r_0).$$

Similarly, we have

$$N_{\Psi}^{\infty}(r, r_0) \leq \triangle n(l-1)N_{\chi}^{0}(r, r_0).$$

On the other hand, we have $\|\chi\| \leq 2\|f\|\|g\|$. It then follows that outside of an analytic set of codimension ≥ 2, the functions $\tilde{\phi}\chi^{\triangle n(l-1)}$ and $\tilde{\psi}\chi^{\triangle n(l-1)}$ are both holomorphic on $B(1)$.

Set

$$t := \frac{\rho}{dq\triangle - lN - 2\triangle n(l-1)}$$

and define a plurisubharmonic function u by

$$u := t \log |\tilde{\phi}\tilde{\psi}\chi^{2\triangle n(l-1)}|.$$

Since $\rho + 2 \triangle tn(l-1) = t(dq \triangle - lN)$ we obtain that

$$\begin{aligned}
\det(h_{i\bar{j}})e^{u+u_1+u_2} &\leq |\tilde{\phi}|^t|\tilde{\psi}|^t|\chi|^{2\triangle tn(l-1)}\|f\|^{\rho}\|g\|^{\rho} \\
&\leq K|\tilde{\phi}|^t|\tilde{\psi}|^t\|f\|^{\rho+2\triangle tn(l-1)}\|g\|^{\rho+2\triangle tn(l-1)} \\
&\leq K|\tilde{\phi}|^t|\tilde{\psi}|^t\|f\|^{t(dq\triangle-lN)}\|g\|^{t(dq\triangle-lN)}
\end{aligned}$$

for some constant K. The volume form on M is given by

$$dV := c_m \det(h_{i\bar{j}})v_m.$$

Therefore,

$$\begin{aligned}
I :&= \int_{B(1)} e^{u+u_1+u_2}dV \\
&\leq K \int_{B(1)} |\tilde{\phi}|^t\|f\|^{t(dq\triangle-lN)}|\tilde{\psi}|^t\|g\|^{t(dq\triangle-lN)}v_m
\end{aligned}$$

where K is some positive constant.

Let $p_1 = p_2 = 2$; then $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and by Hölder inequality we have:

$$\begin{aligned}
I &\leq K\Big(\int_{B(1)} |\tilde{\phi}|^{tp_1}\|f\|^{tp_1(dq\triangle+lN)}v_m\Big)^{1/p_1}\Big(\int_{B(1)} |\tilde{\psi}|^{tp_2}\|g\|^{tp_2(dq\triangle-lN)}v_m\Big)^{1/p_2} \\
&\leq K\Big(\int_0^1 r^{2m-1}\big(\int_{S(r)} |\tilde{\phi}|^{tp_1}\|f\|^{tp_1(dq\triangle+lN)}\sigma_m\big)dr\Big)^{1/p_1} \times \\
&\Big(\int_0^1 r^{2m-1}\big(\int_{S(r)} |\tilde{\psi}|^{tp_2}\|g\|^{tp_2(dq\triangle-lN)}\sigma_m\big)dr\Big)^{1/p_2}.
\end{aligned}$$

Using Lemma 2.1.5 with $\epsilon = d$, we have

$$-\frac{lN}{\triangle} \geq -d(n+1) - d/2.$$

So

$$\begin{aligned}
dq \triangle -lN - 2 \triangle n(l-1) &= \triangle\Big(dq - \frac{lN}{\triangle} - 2n(l-1)\Big) \\
&\geq \triangle\Big(dq - d(n+1) - \frac{d}{2} - 2n(l-1)\Big) \\
&= d \triangle \Big(q - (n+1) - \frac{1}{2} - \frac{2n(l-1)}{d}\Big) \\
&\geq d \triangle \Big(\frac{\rho l(l-1)}{d}\Big) \\
&= \triangle \rho l(l-1).
\end{aligned}$$

So

$$\begin{aligned}
tp_2 l(l-1)/2 = tp_1 l(l-1)/2 = tl(l-1) &= \frac{l(l-1)\rho}{dq \triangle -lN - 2 \triangle n(l-1)} \\
&\leq \frac{1}{\triangle} \\
&< 1.
\end{aligned}$$

Take some p' with $0 < tl(l-1) < p' < 1$. By the same argument as in the proof of Theorem 2.1.6, it follows from Proposition 1.3.16 that for $r_0 < r < R < R_0$,

$$\begin{aligned}
\int_{S(r)} |\tilde{\phi}|^{p_1 t} \|f\|^{p_1 t(dq\triangle - LN)} \sigma_m &= \int_{S(r)} (|\tilde{\phi}| \|f\|^{(dq\triangle - LN)})^{p_1 t} \sigma_m \\
&= \int_{S(r)} |z^{\alpha^1 + \ldots + \alpha^{n+1}} \frac{W_\alpha(F) \|f\|^{(dq\triangle - LN)}}{Q_1^{\triangle}(f) \ldots Q_q^{\triangle}(f)}|^{p_1 t} \sigma_m \\
&\leq K_3 \Big(\frac{1}{R-r} T_f(R, r_0)\Big)^{p'}.
\end{aligned}$$

Likewise, we have

$$\int_{S(r)} |\tilde{\psi}|^{p_2 t} \|g\|^{p_2 t(dq\triangle - LN)} \sigma_m \leq K_4 \Big(\frac{1}{R-r} T_g(R, r_0)\Big)^{p'}.$$

We can conclude that

$$\int_{B(1)} e^{u + u_1 + u_2} dV < \infty.$$

On the other hand, by the result of Yau ([20]) and Karp ([22]), we have

$$\int_{B(1)} e^{u+u_1+u_2} dV = \infty,$$

because $u + u_1 + u_2$ is plurisubharmonic. This is a contradiction.

We now deal with the case

$$\lim_{r \longrightarrow 1} \sup \frac{T_f(r, r_0)}{\log 1/(1-r)} = \infty.$$

Now, applying theorem 2.1.8 with $\epsilon = 1/2$ to the maps f and g, we get

$$(q - (n+1) - 1/2) T_f(r, r_0) \leq \sum_{j=1}^{q} d^{-1} N_f^{[l-1]}(r, D_j) + S_f(r),$$

$$(q - (n+1) - 1/2) T_g(r, r_0) \leq \sum_{j=1}^{q} d^{-1} N_g^{[l-1]}(r, D_j) + S_g(r),$$

where $S_f(r)$ and $S_g(r)$ are giving like in theorem 2.1.8. Adding these two inequalities give

$$\Big(q - (n+1) - 1/2\Big)\Big(T_f(r_0, r) + T_g(r_0, r)\Big) \leq \frac{1}{d} \sum_{j=1}^{q} (N_f^{[l-1]}(r, D_j) + N_g^{[l-1]}(r, D_j))$$
$$S_g(r) + S_f(r).$$

Using the fact that the D_js are in general position, the sum $\sum_{j=1}^{q} (N_f^{[l-1]}(r, D_j) + N_g^{[l-1]}(r, D_j))$ counts each point of the set $A = \cup_{j=1}^{q} (f^{-1}(D_j) \cup g^{-1}(D_j))$ with order at most $2n(l-1)$. Hence,

$$\Big(q - (n+1) - 1/2\Big)\Big(T_f(r_0, r) + T_g(r_0, r)\Big) \leq \frac{2n(l-1)}{d} N(r, A) + S_g(r) + S_f(r). \quad (2.28)$$

We now claim that

$$N(r, A) \leq T_f(r_0, r) + T_g(r_0, r) + O(1).$$

Indeed, consider the map $\chi := f_{i_0}g_{j_0} - f_{j_0}g_{i_0}$ defined in (2.27). If $z \in A$, then $f(z) = g(z)$ and so $\chi(z) = 0$. It then follows that $N(r, A) \leq N^0_\chi(r, r_0)$. By the first main theorem,

$$N^0_\chi(r, r_0) \leq T_\chi(r, r_0) + O(1) \leq T_f(r, r_0) + T_g(r, r_0) + O(1),$$

where in the last inequality, we have used the fact $\|\chi\| \leq 2\|f\|\|g\|$. The claim then follows.

(2.28) therefore gives

$$\begin{aligned} \Big(q - (n+1) - 1/2\Big)\Big(T_f(r_0, r) + T_g(r_0, r)\Big) &\leq \\ \frac{2n(l-1)}{d}(T_f(r) + T_g(r)) + S_g(r) + S_f(r) &\leq \\ \Big(\frac{2n(l-1)}{d} + \frac{\rho l(l-1)}{d}\Big)(T_f(r) + T_g(r)) + S_g(r) + S_f(r). \end{aligned}$$

So

$$\Big(q - (n+1) - 1/2 - \frac{2n(l-1)}{d} - \frac{\rho l(l-1)}{d}\Big)(T_f(r_0, r) + T_g(r_0, r)) \leq S_g(r) + S_f(r).$$

(2.28) therefore gives

$$\begin{aligned} \Big(q - (n+1) - 1/2\Big)\Big(T_f(r_0, r) + T_g(r_0, r)\Big) &\leq \\ \frac{2n(l-1)}{d}(T_f(r) + T_g(r)) + S_g(r) + S_f(r) &\leq \\ \Big(\frac{2n(l-1)}{d} + \frac{\rho l(l-1)}{d}\Big)(T_f(r) + T_g(r)) + S_g(r) + S_f(r). \end{aligned}$$

So

$$\Big(q - (n+1) - 1/2 - \frac{2n(l-1)}{d} - \frac{\rho l(l-1)}{d}\Big)(T_f(r_0, r) + T_g(r_0, r)) \leq S_g(r) + S_f(r).$$

If $R_0 = \infty$, then (2.28) leads to a contradiction since

$$\lim_{r \longrightarrow \infty} \frac{S_f(r) + S_g(r)}{T_f(r_0, r) + T_g(r_0, r)} = 0.$$

In the case $R_0 < \infty$, if we assume

$$\limsup_{r \to R_0} \frac{T_f(r, r_0)}{\log(1/R_0 - r)} = \infty,$$

then, again (2.28) leads to a contradiction. We conclude the proof of Theorem 1.2.3 by observing that $I(1/d) \leq I(2)$, where $I(x) := \min\{k \in \mathbb{N} : k > x\}$.

□

2.3 The Gauss map of a complete regular submanifold of $\mathbb{C}^m$

For a general oriented $k-$submanifold of $\mathbb{R}^n$ the Gauss map can be defined, and its target space is the oriented Grassmannian $\tilde{G}_{k,n}$, i.e. the set of all oriented $k-$planes in $\mathbb{R}^n$. In this case a point on the submanifold is mapped to its oriented tangent subspace. One can also map to its oriented normal subspace; these are equivalent as $\tilde{G}_{k,n} \cong \tilde{G}_{n-k,n}$ via orthogonal complement. In Euclidean $3-$space, this says that an oriented 2-plane is characterized by an oriented 1-line, equivalently a unit normal vector (as $\tilde{G}_{1,n} \cong S^{n-1}$), hence this is consistent with the definition above.

Finally, the notion of Gauss map can be generalized to an oriented submanifold X of dimension k in an oriented ambient Riemannian manifold M of dimension n. In that case, the Gauss map then goes from X to the set of tangent $k-$planes in the tangent bundle TM. The target space for the Gauss map N is a Grassmann bundle built on the tangent bundle TM. In the case where $M = \mathbb{R}^n$, the tangent bundle is trivialized (so the Grassmann bundle becomes a map to the Grassmannian), and we recover the previous definition.

Let $f = (f_1, ..., f_m) : M \to \mathbb{C}^m$ be a regular submanifold of $\mathbb{C}^m$, namely, M be a connected complex manifold and f be a holomorphic map of M into $\mathbb{C}^m$ such that rank

$d_p f = \dim M$ for every point $p \in M$.

To each point $p \in M$, we assign the tangent space $T_p(M)$ of M at p which may be regarded as an $n-$dimensional linear subspace of $T_{f(p)}\mathbb{C}^m$. On the other hand, each $T_p(\mathbb{C}^m)$ is identify with $T_0(\mathbb{C}^m) = \mathbb{C}^m$ by a parallel translation. Therefore, to each $T_p(M)$ corresponds a point $G(p)$ in the complex Grassmannian manifold $G(n, m)$ of all $n-$dimensional linear subspaces of $\mathbb{C}^m$, where $n = \dim M$.

Definition 2.3.1 *We call the map $G : M \longrightarrow G(n, m)$ the Gauss map of $f : M \longrightarrow \mathbb{C}^m$.*

The space $G(n, m)$ is canonically embedded in $\mathbb{P}^N(\mathbb{C}) = \mathbb{P}(\wedge^n \mathbb{C}^m)$, where $N = \begin{pmatrix} m \\ n \end{pmatrix} - 1$. The Gauss map G may be identified with holomorphic map of M into $\mathbb{P}^N(\mathbb{C})$ given as follows:
Taking holomorphic local coordinates $(z_1, ..., z_n)$ defined on an open set U, we consider the map

$$\bigwedge := D_1 f \wedge \cdots \wedge D_n f : U \longrightarrow \bigwedge^n \mathbb{C}^m - \{0\},$$

where $D_i f = ((\partial/\partial z_i) f_1, \cdots, (\partial/\partial z_i) f_{N+1})$. Then,

$$G = \pi . \bigwedge$$

locally, where $\pi : \mathbb{C}^{N+1} - \{0\} \longrightarrow \mathbb{P}^N(\mathbb{C})$ is the canonical projection map.

A regular submanifold M of $\mathbb{C}^m$ is considered a Kähler manifold with the metric ω induced from the standard flat metric on $\mathbb{C}^m$. By dV we denote the volume form on M. We can see that For arbitrarily holomorphic coordinates $z_1, ..., z_n$,

$$dV = |\bigwedge|^2 \Big(\frac{\sqrt{-1}}{2}\Big)^n dz_1 \wedge d\bar{z_1} \wedge \cdots \wedge dz_n \wedge d\bar{z_n},$$

where

$$|\bigwedge|^2 = \sum_{1\le i_1<\cdots<i_n\le m} \Big|\frac{\partial(f_{i_1},...,f_{i_n})}{\partial(z_1,...,z_n)}\Big|^2.$$

For a regular submanifold $f : M \longrightarrow \mathbb{C}^m$ the Gauss map $G : M \longrightarrow \mathbb{P}^N(\mathbb{C})$ satisfies condition

$$\Omega_G + dd^c \log h^2 = dd^c \log |\bigwedge|^2 = \text{Ric } (\omega)$$

where $h = 1$.

As a direct consequence of Theorem 2.1.6 , we have

Theorem 2.3.2 *Let $f : M \to \mathbb{C}^m$ be a complete regular submanifold such that the universal covering of M is biholomorphic to $B(R_0)$ $(0 < R_0 \le +\infty)$. If the Gauss map $G : M \to \mathbb{P}^N(\mathbb{C})$ is algebraically nondegenerate, then for every hypersurfaces $D_1, ..., D_q$ of degree d_j $j = 1, ..., q$ in general position, by letting $d =$l.c.m. $\{d_1, \dots, d_q\}$ (the least common multiple of $\{d_1, \dots, d_q\}$), we have, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \delta^G_{l-1}(D_j) \le N + 1 + \epsilon + \frac{l(l-1)}{d}$$

where $l \le 2^{N^2+4N} e^N d^{2N} (NI(\epsilon^{-1}))^N$, $n = \dim M$, and $N = \begin{pmatrix} m \\ n \end{pmatrix} - 1$.

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