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RANK ONE DYNAMICS NEAR HETEROCLINIC CYCLES AND CONDITIONAL MEMORY LOSS FOR NONEQUILIBRIUM DYNAMICAL SYSTEMS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Anushaya Mohapatra

August 2013

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Abstract

There are two parts in this dissertation. In the first part we prove that genuine nonuniformly hyperbolic dynamics emerge when flows in $\mathbb{R}^N$ with homoclinic loops or heteroclinic cycles are subjected to certain time-periodic forcing. In particular, we establish the emergence of strange attractors and SRB measures with strong statistical properties (central limit theorem, exponential decay of correlations, *et cetera*). We identify and study the mechanism responsible for the nonuniform hyperbolicity: saddle point shear. Our results apply to concrete systems of interest in the biological and physical sciences, such as May-Leonard models of Lotka-Volterra dynamics.

In the second part we introduce a notion of conditional memory loss for nonequilibrium open dynamical systems. We prove that this type of memory loss occurs at an exponential rate for nonequilibrium open systems generated by one-dimensional piecewise-differentiable expanding Lasota-Yorke maps.

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CHAPTER 1

Introduction

The nonuniform hyperbolicity theory is rich with interesting ideas and techniques. This theory provides a rigorous mathematical foundation for the phenomenon known as deterministic chaos and applies broadly in physics, engineering, neuroscience, and biology.

Originating with the works of Lyapunov [56] and Perron [71], the nonuniform hyperbolicity theory has emerged from the work of Pesin [72] as a relatively well-understood mathematical theory in dynamical systems. Using nonzero Lyapunov exponents as a point of departure, Pesin theory describes the ergodic properties of smooth dynamical systems that admit invariant hyperbolic measures. The first part of this dissertation addresses the following challenges.

(1) What mechanisms produce nonuniformly hyperbolic dynamics in physical/biological systems?

(2) Can one prove the existence of genuine nonuniformly hyperbolic dynamics in concrete systems?

Until the early 1990s, no genuinely nonuniformly hyperbolic examples were provably known. Sinai/Ruelle/Bowen (SRB) measures were first constructed in a genuinely nonuniformly hyperbolic context for the Hénon family by Bendicks and Young [11]. Building on this work and on tower techniques of Young([89], [90]), Wang and Young developed a comprehensive theory for families of dissipative diffeomorphisms with one direction of instability.

Questions (1) and (2) have been addressed in the contexts of periodically-kicked limit cycles and intermittent maps. Using the rank one theory of Wang and Young, it is possible to prove that if shear exists in a neighborhood of the limit cycle, nonuniformly hyperbolic dynamics may emerge when the system is kicked periodically [66, 86]. Lin and Young call this mechanism shear-induced chaos [50].

Here we consider smooth flows in any physical dimension N with homoclinic orbits or heteroclinic cycles. We assume that the flow is dissipative near each saddle. We identify a mechanism we call saddle point shear and we prove that saddle point shear can produce nonuniform hyperbolicity when such flows are periodically forced. Further, we prove the existence of SRB measures with strong statistical properties, including a central limit theorem and exponential decay of correlations.

Now we briefly describe the relation of our work with existing results in this area and its contributions. As indicated above, Wang and Young studied periodically-forced limit cycles with pulsatile forcing functions and Ott and Wang studied two dimensional systems with homoclinic orbits and with a specific trigonometric forcing function [83] in this context. Our settings is completely general and works for any C^4 forcing functions and near heteroclinic cycles in any dimension. Our results can be applied to any physical systems with a dissipative heteroclinic cycle or homoclinic orbit. Heteroclinic cycles occur frequently in models for ecological dynamics, fluid mechanical instabilities, mathematical biology and game theory [1, 32, 39, 5]. Intermittent behavior in dynamical processes may be described with the help of heteroclinic cycles. Our results applies to many physical and biological model and provides a tools to establish the existence of a comprehensive nonuniformly hyperbolic dynamical profile. In addition, our results lay the foundation for studies that combine analytic and numerical methods. While heteroclinic phenomena often occur in systems with symmetries, our results are independent of symmetry considerations.

In the past dissipative systems received only limited attention in classical mechanics. This is because it was believed that all orbits in these systems eventually either go to fixed points or periodic orbits. Subsequent research would show that the situation is more complex. Van der Pol first studied a periodically-forced damped nonlinear oscillator and showed that it can have interesting behavior. Cartwright and Littlewood proved later that in certain parameter ranges, the Van der Pol oscillator has periodic orbits of different periods [19]. A number of other differential equations with chaotic behavior have been studied in the last few decades, both numerically and analytically. In the dissipative setting examples include the equations of Lorenz [54, 77, 75], the Duffing equation [33], Lorentz gases acted on by external forces and modified Van der Pol-type systems [47].

1.1 Nonequilibrium open dynamical systems

In the second part of this dissertation we study the dynamics of nonequilibrium open dynamical systems. By nonequilibrium, we mean that the dynamical model itself may vary with time. By open, we mean that the phase space contain holes through which trajectories may escape. We are motivated here by dynamical processes that evolve in time varying environments or that contain time-varying parameters. For example, consider billiard systems wherein both the scatterers and holes may move. Such models are of interest in quantum optics, acoustic chemical dynamics, astronomy, and experimental study of electrons in semiconductors [38, 28].

Since chaotic dynamical systems exhibit sensitive dependence on initial conditions, it is natural to take a statistical point of view when studying their long-term behavior. When studying iterates of a single map, one may start with a particular class of initial probability measures, evolve these probability measures, and look for limiting invariant measures. However for time-dependent (nonequilibrium) dynamical systems without holes, invariant measures will not exist in general. For this reason, we study the evolution of pairs of probability measures. We say that a nonequilibrium system without holes exhibits statistical memory loss if for any two suitable initial probability measures, the distance between them converges to zero in a suitable metric as they evolve. For an autonomous system, statistical memory loss is equivalent to decay of correlations.

Ott, Stenlund, and Young studied memory loss for nonequilibrium expanding maps in any dimension and piecewise expanding maps in dimension one. They prove that such systems lose memory in the statistical sense at an exponential rate [68]. Memory loss for time-dependent piecewise-expanding systems in higher dimensions is studied by Gupta, Ott, and Török [30]. They prove statistical memory loss at an exponential rate using oscillation norm.

The theory of open dynamical systems is much less developed than that of closed systems. Open systems have been studied in the contexts of of escape rates and conditionally invariant measures [6, 16, 21, 23]. In the second part of dissertation we study statistical properties of open and nonequilibrium dynamical systems. We do so by introducing a notion of conditional memory loss in statistical sense. We prove this type of conditional memory loss occurs in a certain one-dimensional setting. When studying memory loss or the related problems of decay to equilibrium/decay of correlations, one may employ a number of techniques, including spectral methods, coupling methods, and the use of convex cones and the Hilbert metric. We prove our result by using convex cones and Hilbert metric.

1.2 Dissertation overview

In the first part of dissertation, we study rank one dynamics near homoclinic orbits and heteroclinic cycles. In second part, we study conditional memory loss for nonequilibrium open dynamical systems. The first part includes Chapters 2-4 and the second part includes Chapters 5 and 6. In Chapter 2 we provide background for our heteroclinic cycles results and we discuss saddle point shear mechanism. In Chapter 3 we state and prove our main results on dynamics near heteroclinic cycles and homoclinic orbits. Chapter 4 contains the proofs of several technical results stated in Chapter 3. Chapter 5 contain background information on nonequilibrium open dynamical systems. In Chapter 6 we state and prove our main result on statistical memory loss for nonequilibrium open dynamical systems. Finally we discuss future directions in Chapter 7.

CHAPTER 2

Background and Motivation

Smooth dynamics is the study of differentiable flows or maps, and in these situations one may try to develop local information from the infinitesimal information provided by the differential. Among smooth dynamical systems, hyperbolic dynamics is characterized by the presence of expanding and contracting directions for the derivative. The study of hyperbolic dynamics began with the study of uniformily hyperbolic dynamics. Introduced by Smale, a uniformly hyperbolic set associated with a smooth map is one over which the tangent bundle splits into two invariant subbundles, one contracting and one expanding. Uniform hyperbolicity theory has many applications within mathematics, such as to geometry, modern rigidity theory, dimension theory, and statistical and mathematical physics.

Few physical processes have a uniformly hyperbolic character. There are many reasons for this, among them discontinuities and singularities (*e.g.* the Lorentz gas), transient effects, neutral directions, and nonuniform effects. It is common in applications to find some hyperbolic behavior, but without uniformity of contraction and expansion. Nonuniform hyperbolicity theory allows the asymptotic expansion and contraction rates to depend on the point in a way that does not admit uniform bounds which holds on a subset of the space.

2.1 Pesin's theory

Nonuniform hyperbolicity conditions can be expressed in terms of the Lyapunov exponents. Namely, a dynamical system is nonuniformly hyperbolic if it admits an invariant measure with nonzero Lyapunov exponents almost everywhere.

Throughout this section let $f : M \to M$ be a diffeomorphism of a compact smooth Riemannian manifold M. The **Lyapunov exponent** of the vector $\boldsymbol{v} \in T_xM$ at the point $x \in M$ is defined by $\chi^+(x, \boldsymbol{v}) := \lim_{n\to\infty} \frac{1}{n} \log\|D_x f^n(\boldsymbol{v})\|$, if the limit exist. This takes only finitely many values $\chi_1^+(x) < \cdots < \chi_{p^+(x)}^+(x)$ that determine the subspaces $V_i^+(x) := \{\boldsymbol{v} \in T_xM : \chi^+(x, \boldsymbol{v}) \le \chi_i^+(x, \boldsymbol{v})\}$ (which are nested). Similarly, backward Lyapunov exponents $\chi_i^-(x, \boldsymbol{v})$ can be obtained as $n \to -\infty$. Pesin started with an invariant probability measure and described the properties of corresponding ergodic system assuming almost every orbits have nonzero Lyapunov exponents. The following multiplicative ergodic theorem of Oseledets is a key result on the regularity of trajectories.

Theorem 2.1.1. *For any invariant Borel probability measure ν, almost every $x \in M$*

is Lyapunov regular in the following sense:

1. $p^+(x) = p^-(x) =: p(x)$

2. $T_xM = \oplus_{i=1}^{p(x)} E_i(x)$ *where* $E_i(x) := V_i^+(x) \cap V_i^-(x)$

3. $\lim_{n\to\pm\infty} \frac{1}{n} \log\|D_x f^n(\boldsymbol{v})\| = \chi_i^+(x) = -\chi_i^-(x) := \chi_i(x)$

   uniformly in $\{\boldsymbol{v} \in E_i(x) : \|v\| = 1\}$.

A diffeomorphisim f is said to have nonzero exponents on an invariant set Λ if for each $x \in \Lambda$ there is an $s = s(x)$ such that

$$\chi_1(x) < .... < \chi_{s(x)}(x) < 0 < \chi_{s+1}(x) < ... < \chi_{p(x)}(x). \tag{2.1}$$

An invariant Borel probability measure ν on an invariant set Λ is called **hyperbolic measure** if equation (2.1) holds a.e $x \in \Lambda$. Pesin's work on relating Lyaponuv exponent and nonuniform hyperbolicity can be summarized by the following theorem.

Theorem 2.1.2. *Let* Λ *be a f-invariant set and let* ν *be an ergodic hyperbolic probability measure on it. Let* $E^s(x) = \oplus_{i=1}^{s} E_i(x)$ *and* $E^u(x) = \oplus_{i=s+1}^{k} E_i(x)$*.The subspaces* $E^u(x)$ *and* $E^s(x)$ *for* $x \in \Lambda$ *have the following properties:*

L1. $E^u(x)$ *and* $E^s(x)$ *depend measurably on* x*.*

L2. $T_xM = E^s(x) \oplus E^u(x)$*.*

L3. they are invariant $d_xf(E^s(x)) = E^s(f(x))$ *and* $d_xf(E^u(x)) = E^u(f(x))$

L4. there exist $\varepsilon_0 > 0$ *and measurable functions* $C(x, \varepsilon) > 0, K(x, \varepsilon) > 0$*,* f*-invariant Borel functions* $\lambda_1, \lambda_2 : \Lambda \to \mathbb{R}^+$ *with* $\lambda_1 < 1 < \lambda_2$ *and* $0 < \varepsilon \leq \varepsilon_0$ *such that*

   (a) the subspace $E^s(x)$ *is stable:* $\|df_x^n \boldsymbol{v}\| \leq C(x, \varepsilon)(\lambda_1(x))^n \|\boldsymbol{v}\|$
   for $\boldsymbol{v} \in E^s(x)$ *and* $n \in \mathbb{N}$.

(b) the subspace E^u is unstable: $\|df_x^n \boldsymbol{v}\| \leq C(x,\varepsilon)(\lambda_2(x))^{-n}\|\boldsymbol{v}\|$ *for* $\boldsymbol{v} \in E^u(x)$ *and* $n \in \mathbb{N}$.

(c) $\angle(E^s(x), E^u(x)) \geq K(x,\varepsilon)$

(d) $C(f^n(x)) \leq C(x,\varepsilon)e^{|n|}$ *and* $K(f^n(x)) \geq K(x,\varepsilon)e^{-|n|}$ *for* $n \in \mathbb{Z}$.

We summarize this by saying that for any hyperbolic measure ν, the set of Lyapunov regular points with nonzero Lyapunov exponents contains a nonuniformly hyperbolic set of full ν measure. In fact, finding trajectories with nonzero Lyapunov exponents is a universal approach for establishing nonuniform hyperbolicity.
Pesin theory assumes the existence of a hyperbolic measure and proceeds from this starting point. Which dynamical systems admit hyperbolic measures ?

2.2 SRB measure

When a dynamical system possesses some degree of hyperbolicity, individual orbits are typically unstable. Utilizing a probabilistic point of view often yields insight. The following questions are fundamental.

(Q1) Does the dynamical system admit an invariant measure that describes the asymptotic distribution of a large set (positive Riemannian volume) of orbits? If so, is this measure unique?

(Q2) What are the geometric and ergodic properties of the invariant measure(s)? For example, is a central limit theorem satisfied? At what rate do correlations decay? Large deviation principle? Weak or almost-sure invariance principle (approximation by Brownian motion)?

The Birkhoff ergodic theorem applies directly to a conservative system; inparticular to a system preserving a measure μ that is equivalent to Riemannian volume. If μ is ergodic, then almost every orbit with respect to μ and therefore with respect to Riemannian volume is asymptotically distributed according to μ. By contrast, invariant measures associated with dissipative (volume-contracting) systems are necessarily singular with respect to Riemannian volume. Direct application of the Birkhoff ergodic theorem yields no information about (Q1) in the dissipative context. Question (Q1) remains a major challenge.

It is natural in the dissipative context to focus on special invariant sets on which the core dynamics evolve: attractors. Let M be a compact Riemannian manifold and let $F : M \to M$ be a C^2 embedding. A compact set Ω satisfying $F(\Omega) = \Omega$ is called an ***attractor*** if there exists an open set U called its ***basin*** such that $F^n(x) \to \Omega$ as $n \to \infty$ for every $x \in U$. The attractor Ω is said to be

(a) ***irreducible*** if it cannot be written as a union of two disjoint attractors;

(b) ***uniformly hyperbolic*** if the tangent bundle over Ω splits into two DF-invariant subbundles E^s and E^u such that $DF|E^s$ is uniformly contracting, E^u is nontrivial, and $DF|E^u$ is uniformly expanding.

The geometry and ergodic theory of uniformly hyperbolic discrete-time systems are well-understood. In particular, an irreducible, uniformly hyperbolic attractor Ω supports a unique F-invariant Borel probability measure ν with the following property: there exists a set $S \subset U$ with full Riemannian volume in U such that for every continuous observable $\varphi : U \to \mathbb{R}$ and for every $x \in S$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(F^i(x)) = \int_M \varphi \, \mathrm{d}\nu. \tag{2.2}$$

The measure ν is known as a Sinai/Ruelle/Bowen measure (SRB measure). It is natural to link sets of positive Riemannian volume with observable events. If we do so, then the SRB measure ν is observable because temporal and spatial averages coincide for a set of initial data of full Riemannian volume in the basin. SRB measures were first shown to exist for uniformly hyperbolic attractors and the main result in the uniformly hyperbolic context is the following.

Theorem 2.2.1. *[92] Let f be a C^2 diffeomorphism with an uniformly hyperbolic attractor Λ. Then there is a unique f-invariant Borel probability measure μ on Λ that is characterized by each of the following equivalent conditions:*

1. *μ has absolutely continuous conditional measures on unstable manifolds;*

2. *The metric entropy $h_\mu(f)$ is given by*

$$h_\mu(f) = \int \log |det(Df|E^u)| \mathrm{d}\mu$$

3. *There is a set $V \subset U$ of full Riemannian volume such that for every continuous observable $\phi : U \to \mathbb{R}$, we have for every $x \in V$*

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) \to \int \phi \ \mathrm{d}\mu$$

There are analogous results for flows.

SRB measures have their origins in statistical mechanics. The concept of SRB measure has evolved as the theory of nonuniform hyperbolicity has developed. The following definition is state of the art.

Definition 2.2.2. Let M be a compact Riemannian manifold and let $F : M \to M$ be a C^2 embedding. An F-invariant Borel proability measure ν is called an ***SRB measure*** if (F, ν) has a positive Lyapunov exponent ν almost everywhere and if ν has absolutely continuous conditional measures on unstable manifolds.

The following result of Pesin characterizes SRB measures using the metric entropy h_μ.

Theorem 2.2.3. *[7] Let f be an arbitrary diffeomorphisim and μ an f-invariant Borel probability measure with a positive Lyaponuv exponent a.e.Then μ has absolutely continuous conditional measures on W^u if and only if*

$$h_\mu(f) = \int \sum_{\chi_i > 0} \chi_i dim E_i \mathrm{d}\mu$$

where $h_\mu(f)$ is the metric entropy of f with respect μ.

Statistical properties of these measures have been studied using transfer operator methods (*e.g.* [13, 89]), convex cones and projective metrics (*e.g.* [51]), and coupling techniques (*e.g.* [20, 91]).

2.2.1 Construction of SRB measure

SRB measures were first constructed on Axiom A attractors. In the process, it is shown that any limit point of the sequence $\{\frac{1}{n}\sum_{i=0}^{n-1} f_*^i(m_\gamma)\}_{n=1,2\ldots}$ is an SRB measure, where $\gamma \subset \Lambda$ is a piece of local unstable manifold and m_γ denotes the Lebesgue measure on it. The existence and construction result has been extended to the setting in partially hyperbolic setting by Pesin, Viana, and Pollicott [15, 18]. The key tools used are dominated splitting and $\angle(E^u, E^s)$ being bounded away from zero. However these tools cannot be used in the nonuniformly hyperbolic setting. Existence of SRB measure outside uniformly hyperbolic context remains a major challenge.

Hénon first showed by carrying out numerical studies that the family $T_{a,b}(x, y) = (1-ax^2 + y, bx)$, called the Hénon family, has a chaotic attractor for certain parameters. The Hénon family $T_{a,b}$ is a perturbation of logistic family $g_a(x) = 1 - ax^2$. Jakobson proved that for a set of values of 'a' of positive Lebesgue measure g_a has absolutely continuous invariant measure [35]. Benedicks and Carleson studied the Hénon family for small values of b and $a = 2$ [10]. They showed that if $b > 0$ is small enough then for a positive measure set of a-values near $a = 2$ the corresponding diffeomorphism $T_{a,b}$ exhibits a chaotic attractor. SRB measures are constructed for the first time for nonuniformly hyperbolic attractors by Bendicks and Young.

Theorem 2.2.4. *[11] For every $b > 0$ that is sufficiently small, there is a positive Lebesgue measure set $\Delta_b \subset (2 - \varepsilon, 2)$ such that for each $a \in \Delta_b, T_{a,b}$ admits a unique SRB measure.*

Then Wang and Young [84] studied rank-one attractors and developed a theory of SRB measures on these attractor. Identifying mechanisms that produce nonuniform hyperbolicity and proving that nonuniform hyperbolicity is present in concrete models remain major challenges. In this dissertation, we consider these phenomenon in the context of systems with homoclinic loops or heteroclinic cycles. Shear is responsible for emergence of nonuniform hyperbolicity in this context.

2.3 Shear-induced chaos

Recent work has shown that shear is one such mechanism. If a system possesses a substantial amount of intrinsic shear, nonuniform hyperbolicity may be produced when the system is suitably forced. The forcing does not overwhelm the intrinsic dynamics; rather, it acts as an amplifier, engaging the shear to produce nonuniform hyperbolicity. Systems with substantial intrinsic shear may be thought of as excitable systems.

2.3.1 Periodically-kicked limit cycles.

Periodically-kicked limit cycles have received the most attention thus far. We discuss a model of linear shear flow originally studied by Zaslavsky [93]. Consider the following vector field on the cylinder $\mathbb{S}^1 \times \mathbb{R}$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \sigma z \tag{2.3a}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\lambda z. \tag{2.3b}$$

Here $\sigma \geqslant 0$ measures the strength of the angular velocity gradient and $\lambda > 0$ gives the rate of contraction to the limit cycle γ located at $z = 0$. System (2.3) has simple dynamics: every trajectory converges to the limit cycle. However, (2.3) is excitable in a certain parameter regime. The ratio σ/λ measures the amount of intrinsic shear in the system. If this ratio is large, the system is excitable.

Suppose that periodic pulsatile forcing is added to (2.3b), giving

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \sigma z \tag{2.4a}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\lambda z + A\Phi(\theta)\sum_{n=0}^{\infty}\delta(t - nT) \tag{2.4b}$$

Here $A \geqslant 0$ is the amplitude of the forcing, $\Phi : \mathbb{S}^1 \to \mathbb{R}$ is a C^3 function with finitely many nondegenerate critical points, δ is the Dirac delta, and T is the time between kicks (the relaxation time). Figure 2.1 illustrates the dynamics of (2.4). At each time nT, the system receives an instantaneous vertical kick with amplitude A and profile Φ. In particular, the limit cycle γ is deformed into a curve such as the sinusoidal wave depicted in Figure 2.1. After each kick, the system evolves according to (2.3) for T units of time (until the next kick). If both $A\sigma/\lambda$ and T are large, then shear and contraction combine to produce stretch and fold geometry. Figure 2.1 illustrates this geometry: the sinusoidal wave representing the kicked limit cycle morphs into the other curve during the relaxation period.

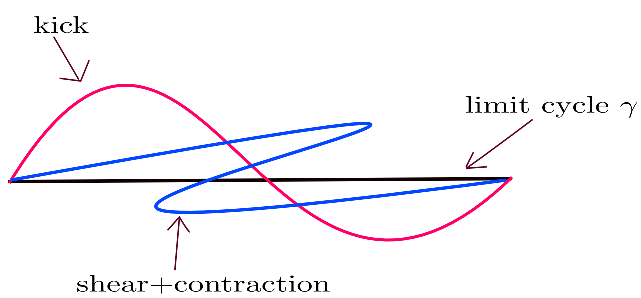

Figure 2.1: Stretch and fold geometry associated with (2.4).

Stretch and fold geometry suggests the presence of SRB measures. It has been shown that (2.4) does produce SRB measures. Wang and Young [86] prove that there exists $C(\Phi) > 0$ such that if $A\sigma/\lambda > C(\Phi)$, then for a set of values of T of positive Lebesgue measure, the time-T map generated by (2.4) has an attractor that supports a unique ergodic SRB measure ν. The dynamics are genuinely nonuniformly hyperbolic and ν has strong statistical properties, among them a central limit theorem and exponential decay of correlations. Wang and Young prove their theorem by applying the theory of rank one maps.

2.4 Theory of rank one maps

The theory of rank one maps [84, 87, 88] provides checkable conditions that imply the existence of nonuniformly hyperbolic dynamics and SRB measures in parametrized families $\{F_a\}$ of dissipative embeddings in dimension N for any $N \geqslant 2$. We give a descriptive summary of the theory and its applications here and a technical description in Section 3.1 of Chapter 3. The term 'rank one' refers to the local character of the embeddings: some instability in one direction and strong contraction in all other directions. Roughly speaking, the theory asserts that under certain checkable conditions, there exists a set Δ of values of a of positive Lebesgue measure such that for $a \in \Delta$, F_a is a genuinely nonuniformly hyperbolic map with an attractor that supports an SRB measure. A comprehensive dynamical profile is given for such F_a; we describe some aspects of this profile now.

The map F_a admits a unique SRB measure ν and ν is mixing. Lebesgue almost every trajectory in the basin of the attractor is asymptotically distributed according to ν and has a positive Lyapunov exponent. Thus the chaos associated with F_a is both observable and sustained in time. The system (F_a, ν) satisfies a central limit theorem, correlations decay at an exponential rate for Hölder observables, and a large deviations principle holds. The source of the nonuniform hyperbolicity is identified and the geometric structure of the attractor is analyzed in detail.

Figure 2.2 illustrates the progression of ideas that has led to the theory of rank one maps. At its core, the theory is based on theoretical developments concerning one-dimensional maps with critical points. We note in particular the parameter exclusion technique of Jakobson [35] and the analysis of the quadratic family by Benedicks and Carleson [9]. The analysis of the Hénon family by Benedicks and Carleson [10] provided a breakthrough from one-dimensional maps with critical points (the quadratic family) to two-dimensional diffeomorphisms (small perturbations of the quadratic family). Mora and Viana [61] generalized the work of Benedicks and Carleson to small perturbations of the Hénon family and proved the existence of Hénon-like attractors in parametrized families of diffeomorphisms that generically unfold a quadratic homoclinic tangency.

$$\boxed{\begin{array}{c}\text{Theory of}\\ 1D\text{ maps}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Hénon maps}\\ \text{and perturbations}\end{array}} \longrightarrow \boxed{\begin{array}{c}\textit{Rank one}\\ \textit{attractors}\end{array}}$$

Figure 2.2: Progression of ideas leading to the theory of rank one maps.

The theory of rank one maps has been applied to many concrete models. The dynamical scenario studied most extensively thus far is that of weakly stable structures subjected to periodic pulsatile forcing. Weakly stable equilibria [66], limit cycles in finite-dimensional systems [67, 85, 86], and supercritical Hopf bifurcations in finite-dimensional systems [86] and infinite-dimensional systems [55] have been

treated. Guckenheimer, Wechselberger, and Young [29] connect the theory of rank one maps and geometric singular perturbation theory by formulating a general technique for proving the existence of chaotic attractors for 3-dimensional vector fields with two time scales. Lin [48] demonstrates how the theory of rank one maps can be combined with sophisticated computational techniques to analyze the response of concrete nonlinear oscillators of interest in biological applications to periodic pulsatile drives. Electronic circuits have been treated as well [64, 65, 81, 82].
We apply this theory to homoclinic and heteroclinic phenomena.

2.5 Homoclinic and heteroclinic phenomena

We study the dynamics near heteroclinic and homoclinic orbits when forced periodically in dimension $N \geq 2$.

Definition 2.5.1. Consider the continuous-time dynamical system described by the ODE $\dot{x} = f(x)$. Suppose there is an equilibrium at $x = x_0$. A solution $\Phi(t)$ is a **homoclinic orbit** if $\Phi(t) \to x_0 \quad \text{as} \quad t \to \pm\infty$. If $f : M \to M$ is a diffeomorphism of a manifold M, we say that x is a homoclinic point if there exists a fixed (or periodic) point p such that $\lim_{n\to\pm\infty} f^n(x) = p$

If the stable and unstable manifolds of a hyperbolic stationary point intersect,they may do so transversely or they may have homoclinic tangencies.
The dynamical picture near transverse intersections is described by the following theorem.

Theorem 2.5.2. *[36] Let M be a smooth manifold, $U \subset M$ open, $f : U \to M$ an*

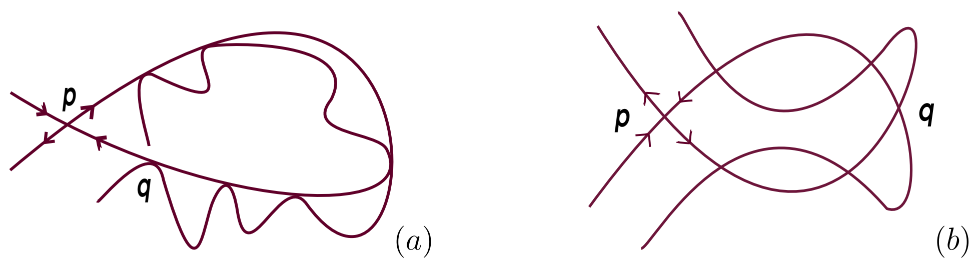

Figure 2.3: (a) homoclinic tangency, (b) transverse homoclinic intersection

embedding, and $p \in U$ a hyperbolic fixed point with transverse homoclinic point q. Then in any arbitrarily small neighborhood of p there exist a horseshoe for some fixed iterate of f. Furthermore the hyperbolic invariant set in the horseshoe contains an iterate of q.

There are significant results concerning the dynamics near homoclinic tangencies of one-parameter families of diffeomarphisms starting from the following significant results by Newhouse.

Theorem 2.5.3. *[62] M is 2-manifold. There is an open subset $U \subset Diff^2(M)$, in which the set of diffeomorphisms exhibiting a homoclinic tangency is dense. It is also implied, in the dissipitive case that there exist a residual subset R of U such that each diffeomorphisim in R has infinitely many hyperbolic periodic attractors(sinks).*

Definition 2.5.4. Let $\phi : M \times \mathbb{R} \to M$ be a C^3 map such that $\phi_\mu(x) = \phi(x, \mu)$ is a diffeomorphisim on M for each $\mu \in \mathbb{R}$. Let $p = p_0$ be a hyperbolic fixed point for ϕ_0 and let q be a homoclinic tangency associated to p. Since p is hyperbolic for small μ we have a unique fixed point p_μ near p and the mapping $\mu \to p_\mu$ is differentiable. Under generic assumptions there are μ-dependent local coordinates such that $W^s(p_\mu)$

is given by $x_2 = 0$ and $W^u(p_\mu)$ by $x_2 = ax_1^2 + b\mu, a \neq 0$ and $b \neq 0$. In this case we say the quadratic homoclinic tangency unfolds generically.

The unfolding of homoclinic tangencies yields a great number of changes in dynamics as μ evolves.

Palis formulated the following conjecture.

Conjecture: Generic one-parameter families of surface diffeomorphisims unfolding a homoclinic tangency exhibit strange attractors or repellers in a persistent way in the measure-theoretic sense (for a positive measure set of values of parameter).

Strange attarctor in this context is a compact invariant set Λ with a dense orbit which has positive Lyaponuv exponents and the stable set $W^s(\Lambda)$ has non-empty interior.

Mora and Viana proved the Palis conjecture for surface diffeomorphisims.

Theorem 2.5.5 ([61])**.** *Let $(f_\mu)_\mu$ be a C^∞ one parameter family of diffeomorphisms on a surface and suppose that f_0 has a homoclinic tangency associated to some periodic point p_0. Then under generic assumptions, there is a positive Lebesgue measure set E of parameter values near $\mu = 0$ such that for each $\mu \in E$ the diffeomorphism f_μ exhibits a strange attractor or repeller near the orbit of tangency.*

2.5.1 Heteroclinic cycles

Definition 2.5.6. A **heteroclinic cycle** is a finite ordered sequence of invariant sets $\{\xi_1, \xi_2, ...\xi_k\}$ and connecting manifolds $\{\Gamma_1, \Gamma_2, ..\Gamma_k\}$ such that Γ_j is backward asymptotic to ξ_j and forward asymptotic ξ_{j+1} with $\xi_{k+1} = \xi_1$. The invariant sets are typically equilibrium points but may include higher dimensional objects such as periodic orbits or chaotic attractors. The connecting manifolds Γ_j are typically isolated trajectory but may be multi-dimensional surfaces. If $k = 1$, the cycle reduce to a homoclinic orbit.

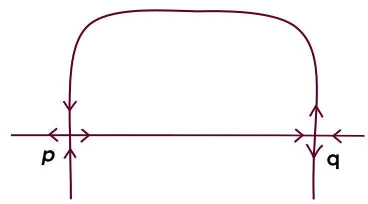

Figure 2.4: heteroclinic cycle with two equilibria p and q

Heteroclinic cycles constitute an important class of solutions. They are associated with intermittent behavior because long pieces of trajectories near the invariant sets ξ_j are followed by the connections Γ_j. We study the effect of external time-periodic forcing and the existence of rank one chaos near heteroclinic cycles of dissipative systems.

Saddle point shear is the mechanism responsible for the emergence of nonuniform hyperbolicity in this context.

2.6 Saddle point shear

We study flows with homoclinic orbits or heteroclinic cycles in dimension $N \geqslant 2$. When a system with a homoclinic orbit is forced with a periodic signal of period T, the stable and unstable manifolds that coincide in the unforced system will typically become distinct. Figure 2.5 illustrates two of the possibilities for the time-T maps. If the stable and unstable manifolds intersect transversely as in Figure 2.5(a), then homoclinic tangles and horseshoes may be produced. The point of intersection may be a point of tangency between the stable and unstable manifolds, a so-called homoclinic tangency. Rich dynamics emerge as a homoclinic tangency is unfolded [24, 45, 61, 84], including the coexistence of infinitely many attracting periodic orbits [62, 63, 69], and nonuniformly hyperbolic horseshoes [70].

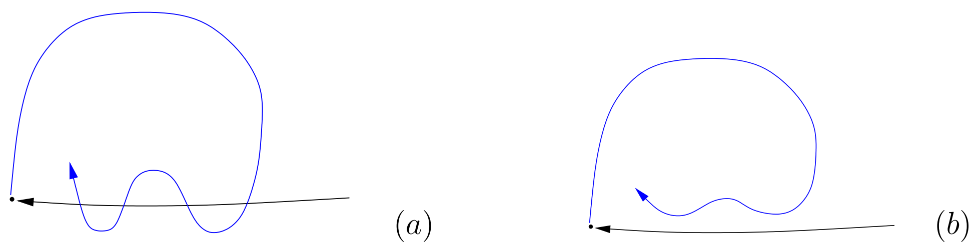

Figure 2.5: Some time-T maps that can occur when a system with a homoclinic loop is subjected to periodic forcing of period T

We focus on the case in which the stable and unstable manifolds of the forced system do not intersect (Figure2.5(b)). Afraimovich and Shilnikov [2] initiated the study of this case by proving that it is possible to define a flow-induced map on a certain cross-section. Our main results concern the dynamics of this flow-induced map. For an unforced flow in any dimension $N \geqslant 2$ with either a homoclinic loop or a heteroclinic cycle, we formulate checkable hypotheses under which a natural map induced by the flow of the forced system admits an attractor that supports a unique ergodic SRB measure for a set of forcing amplitudes μ of positive Lebesgue measure. For such μ, the flow-induced map is rank one in the sense of Wang and Young and therefore the dynamical profile described in [88] applies. In particular, the dynamics are genuinely nonuniformly hyperbolic, a central limit theorem holds, and correlations decay at an exponential rate. Heteroclinic cycles have been studied extensively in connection with dynamics on networks and systems possessing symmetries; see *e.g.* [3, 34, 40, 41]. Figure 3.1 in Chapter 3 illustrates saddle point shear mechanism.

CHAPTER 3

Rank One Chaos near Homoclinic Orbits and Heteroclinic Cycles

We study saddle point shear, a mechanism that can produce sustained, observable chaos in concrete models of physical phenomena. Shear-induced chaos has received substantial recent attention in the context of periodically-kicked limit cycles [50, 55, 67, 85, 86]. We formulate hypotheses that imply the existence of sustained, observable chaos for a set of forcing amplitudes of positive Lebesgue measure. By *sustained, observable chaos* we refer to an array of precisely defined dynamical, geometric, and statistical properties that are made precise in Section 3.1.

3.1 Theory of rank one maps

Let D denote the closed unit disk in $\mathbb{R}^{n-1}$ and let $M = \mathbb{S}^1 \times D$. We consider a family of maps $F_{\boldsymbol{a},b} : M \to M$, where $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathcal{V}$ is a vector of parameters and $b \in B_0$ is a scalar parameter. Here $\mathcal{V} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_k \subset \mathbb{R}^k$ is a product of intervals and $B_0 \subset \mathbb{R} \setminus \{0\}$ is a subset of $\mathbb{R}$ with an accumulation point at 0. Points in M are denoted by (x, y) with $x \in \mathbb{S}^1$ and $y \in D$. Rank one theory postulates the following.

(H1) **Regularity conditions.**

- **(a)** For each $b \in B_0$, the function $(x, y, \boldsymbol{a}) \mapsto F_{\boldsymbol{a},b}(x, y)$ is C^3.
- **(b)** Each map $F_{\boldsymbol{a},b}$ is an embedding of M into itself.
- **(c)** There exists $K_D > 0$ independent of $\boldsymbol{a}$ and b such that for all $\boldsymbol{a} \in \mathcal{V}$, $b \in B_0$, and z, $z' \in M$, we have

$$\frac{|\det DF_{\boldsymbol{a},b}(z)|}{|\det DF_{\boldsymbol{a},b}(z')|} \leqslant K_D.$$

(H2) **Existence of a singular limit.** For $\boldsymbol{a} \in \mathcal{V}$, there exists a map $F_{\boldsymbol{a},0} : M \to \mathbb{S}^1 \times \{0\}$ such that the following holds. For every $(x, y) \in M$ and $\boldsymbol{a} \in \mathcal{V}$, we have

$$\lim_{b \to 0} F_{\boldsymbol{a},b}(x, y) = F_{\boldsymbol{a},0}(x, y)$$

Identifying $\mathbb{S}^1 \times \{0\}$ with $\mathbb{S}^1$, we refer to $F_{\boldsymbol{a},0}$ and the restriction $f_{\boldsymbol{a}} : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $f_{\boldsymbol{a}}(x) = F_{\boldsymbol{a},0}(x, 0)$ as the ***singular limit*** of $F_{\boldsymbol{a},b}$.

(H3) **C^3 convergence to the singular limit.** We select a special index $j \in \{1, \ldots, k\}$. Fix $a_i \in \mathcal{V}_i$ for $i \neq j$. For every such choice of parameters a_i, the maps $(x, y, a_j) \mapsto F_{\boldsymbol{a},b}(x, y)$ converge in the C^3 topology to $(x, y, a_j) \mapsto F_{\boldsymbol{a},0}(x, y)$ on $M \times \mathcal{V}_j$ as $b \to 0$.

(H4) **Existence of a sufficiently expanding map within the singular limit.** There exists $\boldsymbol{a}^* = (a_1^*, \ldots, a_k^*) \in \mathcal{V}$ such that $f_{\boldsymbol{a}^*} \in \mathcal{E}$, where $\mathcal{E}$ is the set of Misiurewicz-type maps defined in Definition 3.1.1 below.

(H5) **Parameter transversality.** Let $C_{\boldsymbol{a}^*}$ denote the critical set of $f_{\boldsymbol{a}^*}$. For $a_j \in \mathcal{V}_j$, define the vector $\tilde{\boldsymbol{a}}_j \in \mathcal{V}$ by $\tilde{\boldsymbol{a}}_j = (a_1^*, \ldots, a_{j-1}^*, a_j, a_{j+1}^*, \ldots, a_k^*)$. We say that the family $\{f_{\boldsymbol{a}}\}$ satisfies the ***parameter transversality*** condition with respect to parameter a_j if the following holds. For each $x \in C_{\boldsymbol{a}^*}$, let $p = f_{\boldsymbol{a}^*}(x)$ and let $x(\tilde{\boldsymbol{a}}_j)$ and $p(\tilde{\boldsymbol{a}}_j)$ denote the continuations of x and p, respectively, as the parameter a_j varies around a_j^*. The point $p(\tilde{\boldsymbol{a}}_j)$ is the unique point such that $p(\tilde{\boldsymbol{a}}_j)$ and p have identical symbolic itineraries under $f_{\tilde{\boldsymbol{a}}_j}$ and $f_{\boldsymbol{a}^*}$, respectively. We have

$$\frac{d}{da_j} f_{\tilde{\boldsymbol{a}}_j}(x(\tilde{\boldsymbol{a}}_j))\bigg|_{a_j = a_j^*} \neq \frac{d}{da_j} p(\tilde{\boldsymbol{a}}_j)\bigg|_{a_j = a_j^*}.$$

(H6) **Nondegeneracy at 'turns'.** For each $x \in C_{\boldsymbol{a}^*}$, there exists $1 \leqslant m \leqslant n-1$ such that

$$\frac{\partial}{\partial y_m} F_{\boldsymbol{a}^*,0}(x,y)\bigg|_{y=0} \neq 0.$$

(H7) **Conditions for mixing.**

(a) We have $e^{\frac{1}{3}\lambda_0} > 2$, where λ_0 is defined within Definition 3.1.1.

(b) Let $J_1, \ldots, J_r$ be the intervals of monotonicity of $f_{\boldsymbol{a}^*}$. Let $Q = (q_{im})$ be the matrix of 'allowed transitions' defined by

$$q_{im} = \begin{cases} 1, & \text{if } f_{\boldsymbol{a}^*}(J_i) \supset J_m, \\ 0, & \text{otherwise.} \end{cases}$$

There exists $N > 0$ such that $Q^N > 0$.

We now define the family $\mathcal{E}$.

Definition 3.1.1. We say that $f \in C^2(\mathbb{S}^1, \mathbb{R})$ is a Misiurewicz map and we write $f \in \mathcal{E}$ if the following hold for some neighborhood U of the critical set $C = C(f) = \{x \in \mathbb{S}^1 : f'(x) = 0\}$.

(1) **(Outside of $\boldsymbol{U}$)** There exist $\lambda_0 > 0$, $M_0 \in \mathbb{Z}^+$, and $0 < d_0 \leqslant 1$ such that

(a) for all $m \geqslant M_0$, if $f^i(x) \notin U$ for $0 \leqslant i \leqslant m-1$, then $|(f^m)'(x)| \geqslant e^{\lambda_0 m}$,

(b) for any $m \in \mathbb{Z}^+$, if $f^i(x) \notin U$ for $0 \leqslant i \leqslant m-1$ and $f^m(x) \in U$, then $|(f^m)'(x)| \geqslant d_0 e^{\lambda_0 m}$.

(2) **(Critical orbits)** For all $c \in C$ and $i > 0$, $f^i(c) \notin U$.

(3) **(Inside $\boldsymbol{U}$)**

(a) We have $f''(x) \neq 0$ for all $x \in U$, and

(b) for all $x \in U \setminus C$, there exists $p_0(x) > 0$ such that $f^i(x) \notin U$ for all $i < p_0(x)$ and $|(f^{p_0(x)})'(x)| \geqslant d_0^{-1} e^{\frac{1}{3}\lambda_0 p_0(x)}$.

The theory of rank one maps states that given a family $\{F_{\boldsymbol{a},b}\}$ satisfying (H1)–(H6), a measure-theoretically significant subset of this family consists of maps admitting attractors with strong chaotic and stochastic properties. We formulate the precise results and we then describe the properties that the attractors possess.

Proposition 3.1.2. *[86] Let $\Phi : \mathbb{S}^1 \to \mathbb{R}$ be a C^3 function with nondegenerate critical points. Then there exist L_1 and δ depending on Φ such that if $L \geq L_1$ and $\Psi : \mathbb{S}^1 \to \mathbb{R}$ is a C^3 function with $\|\Psi\|_{C^2} \leq \delta$ and $\|\Psi\|_{C^3} \leq 1$ then the family*

$$f_a(\theta) = \theta + a + L(\Phi(\theta) + \Psi(\theta)), \;\; a \in [0, 1]$$

satisfies (H4) and (H5). (H7) holds if L_1 is sufficiently large.

Theorem 3.1.3 ([84, 87, 88])**.** *Suppose the family $\{F_{\boldsymbol{a},b}\}$ satisfies (H1), (H2), (H4), and (H6). The following holds for all $1 \leqslant j \leqslant k$ such that the parameter a_j satisfies (H3) and (H5). For all sufficiently small $b \in B_0$, there exists a subset $\Delta_j \subset \mathcal{V}_j$ of positive Lebesgue measure such that for $a_j \in \Delta_j$, $F_{\tilde{\boldsymbol{a}}_j,b}$ admits a strange attractor Ω with properties (P1), (P2), and (P3).*

Theorem 3.1.4 ([84, 85, 87, 88])**.** *In the sense of Theorem 3.1.3,*

$$\textit{(H1)–(H7)} \Longrightarrow \textit{(P1)–(P4)}.$$

Remark 3.1.1. The proof of Theorem 3.1.3 for the special case $n = 2$ appears in [84]. The additional component (H7) $\Rightarrow$ (P4) in Theorem 3.1.4 is proved in [85]. For general n, Wang and Young [87] prove the existence of an SRB measure for $F_{\tilde{\boldsymbol{a}}_j,b}$ if $a_j \in \Delta_j$. The complete proofs of (P1)–(P3) (and (P4) assuming (H7)) for $F_{\tilde{\boldsymbol{a}}_j,b}$ with $a_j \in \Delta_j$ appear in [88] for general n.

We now describe (P1)–(P4) precisely. Write $F = F_{\tilde{\boldsymbol{a}}_j,b}$.

(P1) Positive Lyapunov exponent. Let U denote the basin of attraction of the attractor Ω. This means that U is an open set satisfying $F(\overline{U}) \subset U$ and

$$\Omega = \bigcap_{m=0}^{\infty} F^m(\overline{U}).$$

For almost every $z \in U$ with respect to Lebesgue measure, the orbit of z has a positive Lyapunov exponent. That is,

$$\lim_{m\to\infty} \frac{1}{m} \log \|DF^m(z)\| > 0.$$

(P2) Existence of SRB measures and basin property.

(a) The map F admits at least one and at most finitely many ergodic SRB measures each one of which has no zero Lyapunov exponents. Let $\nu_1, \cdots, \nu_r$ denote these measures.

(b) For Lebesgue-a.e. $z \in U$, there exists $j(z) \in \{1, \ldots, r\}$ such that for every continuous function $\varphi : U \to \mathbb{R}$,

$$\frac{1}{m}\sum_{i=0}^{m-1} \varphi(F^i(x,y)) \to \int \varphi \, \mathrm{d}\nu_{j(z)}.$$

(P3) Statistical properties of dynamical observations.

(a) For every ergodic SRB measure ν and every Hölder continuous function $\varphi : \Omega \to \mathbb{R}$, the sequence $\{\varphi \circ F^i : i \in \mathbb{Z}^+\}$ obeys a central limit theorem. That is, if $\int \varphi \, \mathrm{d}\nu = 0$, then the sequence

$$\frac{1}{\sqrt{m}}\sum_{i=0}^{m-1} \varphi \circ F^i$$

converges in distribution (with respect to ν) to the normal distribution. The variance of the limiting normal distribution is strictly positive unless $\varphi = \psi \circ F - \psi$ for some $\psi \in L^2(\nu)$.

(b) Suppose that for some $L \geqslant 1$, F^L has an SRB measure ν that is mixing. Then given a Hölder exponent η, there exists $\tau = \tau(\eta) < 1$ such that for all Hölder φ, $\psi : \Omega \to \mathbb{R}$ with Hölder exponent η, there exists $K = K(\varphi, \psi)$ such that for all $m \in \mathbb{N}$,

$$\left| \int (\varphi \circ F^{mL})\psi \, \mathrm{d}\nu - \int \varphi \, \mathrm{d}\nu \int \psi \, \mathrm{d}\nu \right| \leqslant K(\varphi, \psi)\tau^m.$$

(P4) Uniqueness of SRB measures and ergodic properties.

(a) The map F admits a unique (and therefore ergodic) SRB measure ν, and

(b) the dynamical system (F, ν) is mixing, or, equivalently, isomorphic to a Bernoulli shift.

3.2 Dynamics near homoclinic loops

Let $N \geqslant 2$ be an integer. Let $\boldsymbol{\xi} = (\xi_i)_{i=1}^N$ denote the standard coordinates in $\mathbb{R}^N$ and let $\{\boldsymbol{e}_i : 1 \leqslant i \leqslant N\}$ be the standard basis for $\mathbb{R}^N$. We start with a C^4 vector field $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^N$ and the associated autonomous differential equation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{\xi}) \tag{3.1}$$

3.2.1 Local dynamical picture

We assume the following dynamical picture in a neighborhood of the origin.

(A1) The origin $\mathbf{0}$ is a stationary point of (3.1) ($\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$). The derivative $D\boldsymbol{f}(\mathbf{0})$ is a diagonal operator with eigenvalues $-\alpha_{N-1} \leq -\alpha_{N-2} \leqslant \cdots \leqslant -\alpha_1 < 0 < \beta$ corresponding to eigenvectors $\boldsymbol{e}_1$ to $\boldsymbol{e}_N$, respectively.

(A2) (dissipative saddle) The eigenvalues of $D\boldsymbol{f}(\mathbf{0})$ satisfy $0 < \beta < \alpha_1$.

(A3) (analytic linearization) There exists a neighborhood U of $\mathbf{0}$ on which $\boldsymbol{f}$ is analytic and on which there exists an analytic coordinate transformation that transforms (3.1) into the linear equation

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}t} = D\boldsymbol{f}(\mathbf{0})\boldsymbol{\eta}.$$

We now add time-periodic forcing to the right side of (3.1). Let $\boldsymbol{p} : \mathbb{R}^N \times \mathbb{S}^1 \to \mathbb{R}^N$ be a C^4 map for which there exists a neighborhood U_2 of $\mathbf{0}$ such that $\boldsymbol{p}$ is analytic on $U_2 \times \mathbb{S}^1$. Adding $\boldsymbol{p}$ to the right side of (3.1) yields the nonautonomous equation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{\xi}) + \mu \boldsymbol{p}(\boldsymbol{\xi}, \omega t), \tag{3.2}$$

where ω is a frequency parameter and μ controls the amplitude of the forcing. We convert (3.2) into an autonomous system on the augmented phase space $\mathbb{R}^N \times \mathbb{S}^1$, giving

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{\xi}) + \mu \boldsymbol{p}(\boldsymbol{\xi}, \theta) \tag{3.3a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega. \tag{3.3b}$$

3.2.2 Two small scales and a useful local coordinate system

Let $\varepsilon_0 > 0$ be such that $\mathcal{U}_{\varepsilon_0} := B(\mathbf{0}, 2\varepsilon_0) \subset U \cap U_2$ and let $\mu_0 > 0$ satisfy $\mu_0 \ll \varepsilon_0$. We focus on forcing amplitudes in the range $[0, \mu_0]$. When the phase space is augmented with an $\mathbb{S}^1$ factor, the hyperbolic saddle $\mathbf{0}$ becomes the hyperbolic periodic orbit $\boldsymbol{\gamma}_0 := \{\mathbf{0}\} \times \mathbb{S}^1$. This hyperbolic periodic orbit persists for μ sufficiently small. Let $\boldsymbol{\gamma}_\mu$ denote the perturbed orbit.

Proposition 3.2.1. *There exists a μ-dependent coordinate system $(\boldsymbol{X}, \theta) = (X_1, \ldots, X_N, \theta)$ defined on $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1$ such that for every $\mu \in [0, \mu_0]$, $\boldsymbol{\gamma}_\mu = \{(\boldsymbol{X}, \theta) : \boldsymbol{X} = \mathbf{0}\}$. That is, we have standardized the location of the hyperbolic periodic orbit. Further, the stable and unstable manifolds $W^s(\boldsymbol{\gamma}_\mu)$ and $W^u(\boldsymbol{\gamma}_\mu)$ are locally flat:*

$$W^s(\boldsymbol{\gamma}_\mu) \cap (\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1) \subset \{(\boldsymbol{X}, \theta) : X_N = 0\}$$

$$W^u(\boldsymbol{\gamma}_\mu) \cap (\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1) \subset \{(\boldsymbol{X}, \theta) : X_i = 0 \text{ for every } 1 \leqslant i \leqslant N-1\}.$$

In terms of $(\boldsymbol{X}, \theta)$ *coordinates, system* (4.2) *on* $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0]$ *has the following form:*

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = (-\alpha_i + \mu G_i(\boldsymbol{X}, \theta; \mu))X_i \qquad (1 \leqslant i \leqslant N-1) \tag{3.4a}$$

$$\frac{\mathrm{d}X_N}{\mathrm{d}t} = (\beta + \mu G_N(\boldsymbol{X}, \theta; \mu))X_N \tag{3.4b}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega. \tag{3.4c}$$

There exists $K_3 > 0$ *such that for each* $1 \leqslant k \leqslant N$, G_k *is analytic on* $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0]$ *and satisfies*

$$\|G_k\|_{C^3(\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0,\mu_0])} \leqslant K_3.$$

Proof of Proposition 3.2.1. The proof of this proposition is given in section (4.1) of Chapter 4. ■

3.2.3 Global dynamical picture

Define the μ-dependent sections Γ^1 and Γ^2 as follows:

$$\Gamma^1 = \{(\boldsymbol{X}, \theta) : X_N = \varepsilon_0,\ \ 0 \leqslant X_1 \leqslant K_0\mu,\ \ -K_0\mu \leqslant X_i \leqslant K_0\mu \text{ for } 2 \leqslant i \leqslant N-1\}$$

$$\Gamma^2 = \left\{(\boldsymbol{X}, \theta) : X_1 = \varepsilon_0,\ \ K_1^{-1}\mu \leqslant X_N \leqslant K_1\mu,\ \ -K_2\mu \leqslant X_i \leqslant K_2\mu \text{ for } 2 \leqslant i \leqslant N-1\right\},$$

where $K_0 > 0$ satisfies $K_0\mu_0 \ll \varepsilon_0$ and $K_1 > 0$ and $K_2 > 0$ are suitably chosen. We assume that for $\mu \in (0, \mu_0]$, the flow generated by (4.2) induces a map from Γ^1 into Γ^2.

(A4) For $\mu \in (0, \mu_0]$, the flow generated by (4.2) induces a C^3 embedding $\mathcal{G}_\mu : \Gamma^1 \to$

Γ^2. Writing $\mathcal{G}_\mu(\boldsymbol{X}, \theta) = (\boldsymbol{Y}, \rho)$, $(\boldsymbol{Y}, \rho)$ has the form

$$Y_1 = \varepsilon_0 \tag{3.5a}$$

$$Y_k = \sum_{i=1}^{N-1} c_{ki} X_i + \mu \Phi_k(X_1, \ldots, X_{N-1}, \theta) \qquad (2 \leqslant k \leqslant N) \tag{3.5b}$$

$$\rho = \theta + \zeta_1 + \mu \Phi_{N+1}(X_1, \ldots, X_{N-1}, \theta). \tag{3.5c}$$

Here (c_{ki}) is an invertible matrix of constants, ζ_1 is a constant, and the functions $\Phi_2, \ldots, \Phi_{N+1}$ are C^3 functions from Γ^1 into $\mathbb{R}$. We assume that $\Phi_N > 0$.

Hypothesis (A4) is motivated by bifurcation scenarios involving homoclinic orbits. Suppose that (3.1) ($\mu = 0$) has a homoclinic solution associated with the saddle $\boldsymbol{X} = \boldsymbol{0}$ that includes the positive X_N component of the local unstable manifold of the saddle and coincides with the positive X_1 axis as $t \to \infty$. (The assumption that the homoclinic orbit coincides with the positive X_1 axis as $t \to \infty$ is not necessary. We proceed in this way to clarify the presentation.) When system (3.1) is forced with a periodic signal ($\mu > 0$), the stable and unstable manifolds will typically break apart. When this happens, transversal intersections may be formed. It is also possible that the stable and unstable manifolds *do not intersect* for $\mu > 0$. In the latter case, it may be possible to define a flow-induced global map from Γ^1 into Γ^2 for $\mu > 0$ sufficiently small. See [83] for an example in which explicit formulas for the global map are derived.

Assuming (A1)–(A4) hold, for $\mu \in (0, \mu_0]$ the flow generated by (4.2) induces a map $\mathcal{M}_\mu : \Gamma^1 \to \Gamma^1$ given by the composition $\mathcal{M}_\mu = \mathcal{L}_\mu \circ \mathcal{G}_\mu$, where $\mathcal{G}_\mu$ is from (A4) and $\mathcal{L}_\mu : \Gamma^2 \to \Gamma^1$ is the 'local' map induced by (4.2). Our primary theorem for systems with homoclinic loops concerns the dynamical properties of the family $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$. Figure 3.1 illustrates the geometry of $\mathcal{M}_\mu$ when $N = 2$.

Theorem 3.2.2. *Assume that system* (4.2) *satisfies (A1)–(A4). Suppose that the* C^3 *function* $\Phi_N(\mathbf{0}, \theta) : \mathbb{S}^1 \to \mathbb{R}$ *has finitely many nondegenerate critical points. Then there exists* $\omega_0 > 0$ *such that for any frequency* ω *satisfying* $|\omega| \geqslant \omega_0$, *there exists a set* $\Delta_\omega \subset (0, \mu_0]$ *of positive Lebesgue measure with the following property. For every* $\mu \in \Delta_\omega$, *the flow-induced map* $\mathcal{M}_\mu$ *admits a strange attractor* Ω *that supports a unique ergodic SRB measure* ν. *The orbit of Lebesgue almost every point on* Γ^1 *has a positive Lyapunov exponent and is asymptotically distributed according to* ν. *The SRB measure* ν *is mixing, satisfies the central limit theorem, and exhibits exponential decay of correlations for Hölder-continuous observables.*

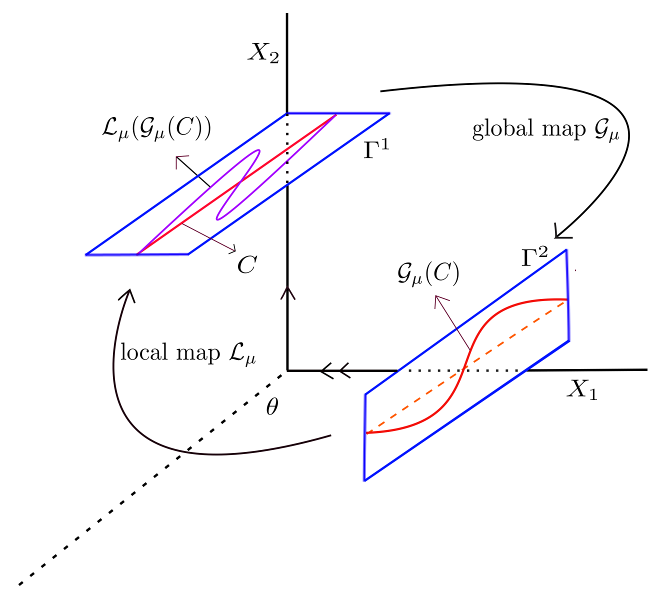

Figure 3.1: Saddle point shear mechanism

The figure 3.1 illustrates the saddle point shear associated with $\mathcal{M}_\mu$.Start with the flat red curve C on Γ^1. Generically, the flow from Γ^1 to Γ^2 will create ripples, meaning that when $\mathcal{G}_\mu(C)$ is viewed as a function of θ, X_2 varies as θ varies. Since the time it takes to travel from Γ^2 to Γ^1 depends on the X_2 coordinate, shear occurs in the θ direction. The purple curve $\mathcal{L}_\mu(\mathcal{G}_\mu(C))$ illustrates the resulting stretch and fold geometry of $\mathcal{M}_\mu$.

3.2.4 Proof of Theorem 3.2.2

The proof of Theorem 3.2.2 requires careful study of the family of flow-induced maps $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$. We will prove that the theory of rank one maps applies to this family. In Section 3.2.5 we compute $\mathcal{L}_\mu$ in a C^3-controlled manner. The $\mu \to 0$ singular limit of the family $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$ is computed in Section 3.2.6. Here we must introduce auxiliary parameters because the direct $\mu \to 0$ limit does not exist. Finally, in Section 3.2.7 we prove that $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$ satisfies the hypotheses of the theory of rank one maps.

Let $p = \ln(\mu^{-1})$. We regard p as the fundamental parameter associated with (4.2). Notice that $\mu \in (0, \mu_0]$ corresponds to $p \in [\ln(\mu_0^{-1}), \infty)$.

We make the coordinate change $(\boldsymbol{X}, \theta) \mapsto (\mu \boldsymbol{x}, \theta)$ on $\mathcal{U}_{\varepsilon_0} \times \mathbb{S}^1$. This stabilizes Γ^1 and Γ^2:

$$\Gamma^1 = \left\{(\boldsymbol{x}, \theta) : x_N = \varepsilon_0 \mu^{-1},\ \ 0 \leqslant x_1 \leqslant K_0,\ \ -K_0 \leqslant x_i \leqslant K_0 \text{ for } 2 \leqslant i \leqslant N-1\right\}$$

$$\Gamma^2 = \left\{(\boldsymbol{x}, \theta) : x_1 = \varepsilon_0 \mu^{-1},\ \ K_1^{-1} \leqslant x_N \leqslant K_1,\ \ -K_2 \leqslant x_i \leqslant K_2 \text{ for } 2 \leqslant i \leqslant N-1\right\}.$$

3.2.5 Computation of $\mathcal{L}_\mu$

We compute $\mathcal{L}_\mu$ in a C^3-controlled manner. We begin by computing a normal form for (4.2) that is valid on $\mathfrak{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0]$. The rescaling $\boldsymbol{X} = \mu \boldsymbol{x}$ transforms (3.4) into

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = (-\alpha_i + \mu g_i(\boldsymbol{x}, \theta; \mu))x_i \qquad (1 \leqslant i \leqslant N-1) \tag{3.6a}$$

$$\frac{\mathrm{d}x_N}{\mathrm{d}t} = (\beta + \mu g_N(\boldsymbol{x}, \theta; \mu))x_N \tag{3.6b}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega, \tag{3.6c}$$

where $g_k(\boldsymbol{x}, \theta; \mu) = G_k(\mu\boldsymbol{x}, \theta; \mu)$ for $1 \leqslant k \leqslant N$. System (3.6) is valid on $\mathcal{D}(\boldsymbol{x}, \theta, \mu) := \mathfrak{U}_{\varepsilon_0} \times \mathbb{S}^1 \times [0, \mu_0]$.

On the time-$t$map induced by (3.6)**.** Let $V(\Gamma^2)$ be a small open neighborhood of Γ^2 in $\mathfrak{U}_{\varepsilon_0} \times \mathbb{S}^1$. We study the time-$t$ map induced by (3.6) assuming that all solutions beginning in $V(\Gamma^2)$ remain inside $\mathfrak{U}_{\varepsilon_0} \times \mathbb{S}^1$ up to time t. Let $\boldsymbol{q}_0 = (\boldsymbol{x}_0, \theta_0) \in V(\Gamma^2)$ and let $\boldsymbol{q}(t, \boldsymbol{q}_0; \mu) = (\boldsymbol{x}(t, \boldsymbol{q}_0; \mu), \theta(t, \boldsymbol{q}_0; \mu))$ denote the solution of (3.6) with $\boldsymbol{q}(0, \boldsymbol{q}_0; \mu) = \boldsymbol{q}_0$. Integrating (3.6), we have

$$x_i(t, \boldsymbol{q}_0; \mu) = x_{i,0} \exp\left(\int_0^t \big(-\alpha_i + \mu g_i(\boldsymbol{q}(s, \boldsymbol{q}_0; \mu); \mu)\big)\,\mathrm{d}s\right) \qquad (1 \leqslant i \leqslant N-1) \tag{3.7a}$$

$$x_N(t, \boldsymbol{q}_0; \mu) = x_{N,0} \exp\left(\int_0^t \big(\beta + \mu g_N(\boldsymbol{q}(s, \boldsymbol{q}_0; \mu); \mu)\big)\,\mathrm{d}s\right) \tag{3.7b}$$

$$\theta(t, \boldsymbol{q}_0; \mu) = \theta_0 + \omega t. \tag{3.7c}$$

We introduce the functions $w_k = w_k(t, \boldsymbol{q}_0; \mu)$ for $1 \leqslant k \leqslant N$ by formulating (3.7) as

$$x_i(t, \boldsymbol{q}_0; \mu) = x_{i,0} \exp\big(t(-\alpha_i + w_i(t, \boldsymbol{q}_0; \mu))\big) \qquad (1 \leqslant i \leqslant N-1) \tag{3.8a}$$

$$x_N(t, \boldsymbol{q}_0; \mu) = x_{N,0} \exp\big(t(\beta + w_N(t, \boldsymbol{q}_0; \mu))\big) \tag{3.8b}$$

$$\theta(t, \boldsymbol{q}_0; \mu) = \theta_0 + \omega t, \tag{3.8c}$$

where

$$w_k(t, \boldsymbol{q}_0; \mu) = \frac{1}{t} \int_0^t \mu g_k(\boldsymbol{q}(s, \boldsymbol{q}_0; \mu); \mu) \, \mathrm{d}s. \tag{3.9}$$

The following proposition establishes C^3 control of the w_k on the domain

$$\mathcal{D}(t, \boldsymbol{q}_0, p) := \left\{(t, \boldsymbol{q}_0, p) : t \in [1, T^*], \;\; \boldsymbol{q}_0 \in V(\Gamma^2), \;\; p \in [\ln(\mu_0^{-1}), \infty)\right\},$$

where T^* is chosen so that all solutions of (3.6) that start in $V(\Gamma^2)$ remain in $\mathfrak{U}_{\varepsilon_0} \times \mathbb{S}^1$ up to time T^*. We view the w_k as functions of t, $\boldsymbol{q}_0$, and p (not μ) for the following estimate.

Proposition 3.2.3. *There exists $K_4 > 0$ such that the following holds. For any $T^* > 1$ such that all solutions of (3.6) that start in $V(\Gamma^2)$ remain in $\mathfrak{U}_{\varepsilon_0} \times \mathbb{S}^1$ up to time T^*, we have*

$$\|w_k\|_{C^3(\mathcal{D}(t, \boldsymbol{q}_0, p))} \leqslant K_4 \mu \qquad (1 \leqslant k \leqslant N).$$

Proof of Proposition 3.2.3. We prove this proposition in Chapter 4. ∎

The stopping time $T(\boldsymbol{q}_0, p)$.

Let $\boldsymbol{q}_0 = (\boldsymbol{x}_0, \theta_0) \in \Gamma^2$ and let $T(\boldsymbol{q}_0, p)$ be the time at which the solution to (3.6) starting from $\boldsymbol{q}_0$ reaches Γ^1. This stopping time is determined implicitly by (3.8b):

$$\varepsilon_0 \mu^{-1} = x_N(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; \mu) = x_{N,0} \exp\big(T(\boldsymbol{q}_0, p) \cdot (\beta + w_N(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; \mu))\big).$$

Solving for T, we have

$$T(\boldsymbol{q}_0, p) = \frac{1}{\beta + w_N(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; \mu)} \ln\left(\frac{\varepsilon_0 \mu^{-1}}{x_{N,0}}\right).$$

The following proposition provides a precise C^3 control of T.

Proposition 3.2.4. *There exists $K_5 > 0$ such that T, viewed as a function of $\boldsymbol{q}_0$ and p, satisfies*

$$\left\| T - \frac{1}{\beta} \ln(\varepsilon_0 \mu^{-1}) \right\|_{C^3(\Gamma^2 \times [\ln(\mu_0^{-1}), \infty))} \leqslant K_5.$$

Proof of Proposition 3.2.4. It is proved in Chapter 4. ■

A C^3-controlled formula for $\mathcal{L}_\mu$.

Let $\boldsymbol{q}_0 = (\boldsymbol{y}, \rho) \in \Gamma^2$ and define $(\boldsymbol{z}, \hat{\theta}) = \mathcal{L}_\mu(\boldsymbol{y}, \rho) := \boldsymbol{q}(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; p)$. We have

$$z_N = \varepsilon_0 \mu^{-1} \tag{3.10a}$$

$$z_i = y_i \left(\frac{\varepsilon_0 \mu^{-1}}{y_N}\right)^{\frac{-\alpha_i + w_i}{\beta + w_N}} \qquad (1 \leqslant i \leqslant N-1) \tag{3.10b}$$

$$\hat{\theta} = \rho + \frac{\omega}{\beta + w_N} \ln\left(\frac{\varepsilon_0 \mu^{-1}}{y_N}\right). \tag{3.10c}$$

3.2.6 The singular limit of $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$

We begin by computing $\mathcal{M}_\mu$. Referring to (A4), the global map $\mathcal{G}_\mu : \Gamma^1 \to \Gamma^2$ is given in rescaled coordinates by $\mathcal{G}_\mu(\boldsymbol{x}, \theta) = (\boldsymbol{y}, \rho)$, where

$$y_1 = \varepsilon_0 \mu^{-1} \tag{3.11a}$$

$$y_i = \sum_{j=1}^{N-1} c_{ij} x_j + \phi_i(x_1, \ldots, x_{N-1}, \theta) \qquad (2 \leqslant i \leqslant N) \tag{3.11b}$$

$$\rho = \theta + \zeta_1 + \mu \phi_{N+1}(x_1, \ldots, x_{N-1}, \theta), \tag{3.11c}$$

Using (3.10) and (3.11),the flow-induced map $\mathcal{M}_\mu$ is given by $\mathcal{M}_\mu(\boldsymbol{x},\theta) = (\boldsymbol{z},\hat{\theta})$

$$z_N = \varepsilon_0\mu^{-1} \tag{3.12a}$$

$$z_1 = \varepsilon_0\mu^{-1}\left(\frac{\varepsilon_0\mu^{-1}}{\sum_{j=1}^{N-1} c_{Nj}x_j + \phi_N}\right)^{\frac{-\alpha_1+w_1}{\beta+w_N}} \tag{3.12b}$$

$$z_i = \left(\sum_{j=1}^{N-1} c_{ij}x_j + \phi_i\right)\left(\frac{\varepsilon_0\mu^{-1}}{\sum_{j=1}^{N-1} c_{Nj}x_j + \phi_N}\right)^{\frac{-\alpha_i+w_i}{\beta+w_N}} \qquad (2 \leqslant i \leqslant N-1) \tag{3.12c}$$

$$\hat{\theta} = \theta + \zeta_1 + \mu\phi_{N+1} + \frac{\omega}{\beta + w_N}\ln\left(\frac{\varepsilon_0\mu^{-1}}{\sum_{j=1}^{N-1} c_{Nj}x_j + \phi_N}\right). \tag{3.12d}$$

We compute the singular limit of $\left\{\mathcal{M}_{\mu(p)} : p \in [\ln(\mu_0^{-1}), \infty)\right\}$ by deriving an auxiliary parameter a from p. This is necessary because the term

$$\frac{\omega}{\beta + w_N}\ln(\varepsilon_0\mu^{-1})$$

in (3.12d) does not converge as $\mu \to 0$. Define $\kappa : (0,\infty) \to \mathbb{R}$ by

$$\kappa(s) = \frac{\omega}{\beta}\ln(s^{-1}).$$

Let $(\mu_n)_{n=1}^\infty$ be any strictly decreasing sequence such that $\mu_n \in (0, \mu_0]$ for all $n \in \mathbb{N}$, $\mu_n \to 0$ as $n \to \infty$, and $\kappa(\mu_n) \in 2\pi\mathbb{Z}$ for all $n \in \mathbb{N}$. For $a \in \mathbb{S}^1$ (here $\mathbb{S}^1$ is identified with $[0, 2\pi)$), define

$$\mu_{a,n} = \kappa^{-1}(\kappa(\mu_n) + a), \qquad p(a,n) = \ln(\mu_{a,n}^{-1}).$$

We now view the family of flow-induced maps as a two-parameter family of embeddings: $\left\{\mathcal{M}_{\mu(p(a,n))} : a \in \mathbb{S}^1,\ n \in \mathbb{N}\right\}$. The parameter n measures the amount of dissipation associated with $\mathcal{M}_{\mu(p(a,n))}$. The following proposition establishes C^3 convergence to a singular limit as $n \to \infty$.

Proposition 3.2.5. *We have*

$$\lim_{n\to\infty} \left\| \mathcal{M}_{\mu(p(a,n))} - (\mathbf{0}, \mathcal{F}_a) \right\|_{C^3(\Gamma^1\times[0,2\pi))} = 0,$$

where $\mathcal{F}_a : \Gamma^1 \to \mathbb{S}^1$ *is given by*

$$\mathcal{F}_a(\boldsymbol{x}, \theta) = \theta + a - \frac{\omega}{\beta} \ln\left(\sum_{j=1}^{N-1} c_{Nj} x_j + \phi_N \right) + \frac{\omega}{\beta} \ln(\varepsilon_0) + \zeta_1. \tag{3.13}$$

Proof of Proposition 3.2.5. We first address the term

$$\frac{\omega}{\beta + w_N} \ln(\mu^{-1})$$

in (3.12d). Decomposing, we have

$$\frac{\omega}{\beta + w_N} \ln(\mu^{-1}) = \frac{\omega}{\beta} \ln(\mu^{-1}) - \frac{\omega w_N}{\beta(\beta + w_N)} \ln(\mu^{-1}). \tag{3.14}$$

Since $\mu = \mu(p(a,n))$, the first term on the right side of (3.14) is equal to a. The asserted C^3 convergence now follows from (A2), Proposition 3.2.3, and Proposition 3.2.4. ■

3.2.7 Verification of the hypotheses of the theory of rank one maps

We show that the family of flow-induced maps $\left\{\mathcal{M}_{\mu(p(a,n))} : a \in \mathbb{S}^1,\ n \in \mathbb{N}\right\}$ satisfies (H1)–(H7). We establish the distortion bound (H1)(c) by studying the families of local maps and global maps separately. Since the matrix (c_{ij}) is invertible, direct computation using (3.11) implies that there exists a distortion constant $D_1 > 0$ such that for every $\mu \in (0, \mu_0]$ and $(\boldsymbol{x}, \theta), (\boldsymbol{x}', \theta') \in \Gamma^1$, we have

$$\frac{|\det D\mathcal{G}_\mu(\boldsymbol{x}, \theta)|}{|\det D\mathcal{G}_\mu(\boldsymbol{x}', \theta')|} \leqslant D_1. \tag{3.15}$$

Now let $(\boldsymbol{y}, \rho) \in \Gamma^2$. Expanding $\det(D\mathcal{L}_\mu(\boldsymbol{y}, \rho))$ via permutations, it follows from (3.10), Proposition 3.2.3, and Proposition 3.2.4 that the leading order term of $\det(D\mathcal{L}_\mu(\boldsymbol{y}, \rho))$ arises from the following combination of derivatives:

$$\partial_{y_N} z_1 \cdot \partial_\rho \hat{\theta} \cdot \prod_{i=2}^{N-1} \partial_{y_i} z_i.$$

It follows by direct computation that there exists $D_2 > 0$ such that for every $\mu \in (0, \mu_0]$ and $(\boldsymbol{y}, \rho), (\boldsymbol{y}', \rho') \in \Gamma^2$, we have

$$\frac{|\det D\mathcal{L}_\mu(\boldsymbol{y}, \rho)|}{|\det D\mathcal{L}_\mu(\boldsymbol{y}', \rho')|} \leqslant D_2. \tag{3.16}$$

Bounds (3.15) and (3.16) imply (H1)(c) with $K_D = D_1 D_2$. Hypotheses (H2) and (H3) follow from Proposition 3.2.5.

Hypotheses (H4), (H5), and (H7) concern the family of circle maps

$$\left\{h_a : \mathbb{S}^1 \to \mathbb{S}^1, \;\; a \in \mathbb{S}^1\right\}$$

defined by

$$h_a(\theta) := \mathcal{F}_a(\mathbf{0}, \theta) = \theta + a - \frac{\omega}{\beta} \ln(\phi_N(\mathbf{0}, \theta)) + \frac{\omega}{\beta} \ln(\varepsilon_0) + \zeta_1.$$

Since $\phi_N(\mathbf{0}, \cdot)$ has finitely many nondegenerate critical points, (H4), (H5), and (H7) follow from proposition (3.1.2) if $|\omega|$ is sufficiently large.

Finally, the nondegeneracy condition (H6) follows by direct computation using (3.13) and the fact that $c_{Nk} \neq 0$ for some $1 \leqslant k \leqslant N-1$. Hence prove the theorem 3.2.2.

3.3 Dynamics near heteroclinic cycles

We start with (3.1) in two dimensions. Set $N = 2$.

3.3.1 Existence of a heteroclinic cycle for the unforced system

We assume that (3.1) has a heteroclinic cycle. The heteroclinic cycle consists of Q_0 hyperbolic saddle equilibria $\{\boldsymbol{q}_i : 1 \leqslant i \leqslant Q_0\}$ and connecting orbits $\{\boldsymbol{\varphi}_i : 1 \leqslant i \leqslant Q_0\}$. Let $-\lambda_i < 0 < \beta_i$ denote the eigenvalues of $D\boldsymbol{f}(\boldsymbol{q}_i)$. The connecting orbits satisfy

$$\lim_{t\to-\infty} \boldsymbol{\varphi}_i(t) = \boldsymbol{q}_i, \qquad \lim_{t\to\infty} \boldsymbol{\varphi}_i(t) = \boldsymbol{q}_{i+1}$$

for $1 \leqslant i \leqslant Q_0 - 1$ and

$$\lim_{t\to-\infty} \boldsymbol{\varphi}_{Q_0}(t) = \boldsymbol{q}_{Q_0}, \qquad \lim_{t\to\infty} \boldsymbol{\varphi}_{Q_0}(t) = \boldsymbol{q}_1.$$

We assume that the saddles satisfy the following hypotheses.

(B1) (dissipative saddles) For each $1 \leqslant i \leqslant Q_0$, the eigenvalues of $D\boldsymbol{f}(\boldsymbol{q}_i)$ satisfy $\lambda_i > \beta_i$.

(B2) (analytic linearizations) For each $1 \leqslant i \leqslant Q_0$, there exists a neighborhood of $\boldsymbol{q}_i$ on which $\boldsymbol{f}$ is analytic and on which there exists an analytic coordinate transformation that transforms (3.1) into its linearization at $\boldsymbol{q}_i$.

As in the homoclinic case, we study system (4.2). Here we assume that $\boldsymbol{p}$ is C^4 on $\mathbb{R}^2 \times \mathbb{S}^1$ and analytic in a neighborhood of each $\{\boldsymbol{q}_i\} \times \mathbb{S}^1$.

When the phase space is augmented with with an $\mathbb{S}^1$ factor, each hyperbolic saddle $\boldsymbol{q}_i$ becomes a hyperbolic periodic orbit $\boldsymbol{\gamma}_{\boldsymbol{q}_i,0} := \{\boldsymbol{q}_i\} \times \mathbb{S}^1$. This hyperbolic periodic orbit persists for μ sufficiently small. Let $\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}$ denote the perturbed orbit. There exists $\varepsilon_0 > 0$ such that for each $1 \leqslant i \leqslant Q_0$, there exists a μ-dependent coordinate system $(\boldsymbol{Z}^{(i)}, \theta) = (Z_1^{(i)}, Z_2^{(i)}, \theta)$ defined on $B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1$ such that for every $\mu \in [0, \mu_0]$, $\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu} = \left\{ (\boldsymbol{Z}^{(i)}, \theta) : \boldsymbol{Z}^{(i)} = \boldsymbol{0} \right\}$ and the stable and unstable manifolds are locally flat:

$$\begin{aligned} W^s(\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}) \cap (B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1) &\subset \left\{ (\boldsymbol{Z}^{(i)}, \theta) : Z_2^{(i)} = 0 \right\} \\ W^u(\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}) \cap (B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1) &\subset \left\{ (\boldsymbol{Z}^{(i)}, \theta) : Z_1^{(i)} = 0 \right\}. \end{aligned}$$

3.3.2 Global dynamical picture

For each $1 \leqslant i \leqslant Q_0$ and $\mu \in (0, \mu_0]$, define the μ-dependent sections S_i and S_i' as follows:

$$\begin{aligned} S_i &= \left\{ (\boldsymbol{Z}^{(i)}, \theta) : Z_1^{(i)} = \varepsilon_0, \;\; C_i^{-1}\mu \leqslant Z_2^{(i)} \leqslant C_i \mu \right\} \\ S_i' &= \left\{ (\boldsymbol{Z}^{(i)}, \theta) : Z_2^{(i)} = \varepsilon_0, \;\; 0 \leqslant Z_1^{(i)} \leqslant C_i' \mu \right\}. \end{aligned}$$

Here the constants C_i' satisfy $C_i'\mu_0 \ll \varepsilon_0$ and the C_i are suitably chosen. We assume that for each $1 \leqslant i \leqslant Q_0$ and $\mu \in (0, \mu_0]$, the flow generated by (4.2) induces a map from S_i' into S_{i+1} (we set $S_{Q_0+1} = S_1$).

(B3) For each $1 \leqslant i \leqslant Q_0$ and $\mu \in (0, \mu_0]$, the flow generated by (4.2) induces a C^3 embedding $\mathcal{G}_{i,\mu} : S_i' \to S_{i+1}$ of the form

$$\mathcal{G}_{i,\mu}(Z_1^{(i)}, \varepsilon_0, \theta) = (\varepsilon_0, b_i Z_1^{(i)} + \mu \Upsilon_i(Z_1^{(i)}, \theta), \theta + \zeta_i + \mu \Psi_i(Z_1^{(i)}, \theta)). \tag{3.17}$$

Here the constants b_i and ζ_i satisfy $b_i \neq 0$ and $\zeta_i \geqslant 0$ for all $1 \leqslant i \leqslant Q_0$. The functions Υ_i and Ψ_i are C^3. We assume that $\Upsilon_i > 0$ for all $1 \leqslant i \leqslant Q_0$.

Figure 3.2 illustrates a sample configuration with 4 saddle equilibria.

Assuming (B1)–(B3) hold, for $\mu \in (0, \mu_0]$ the flow generated by (4.2) induces a map $\mathcal{M}_\mu : S_1' \to S_1'$ given by the composition

$$\mathcal{M}_\mu = (\mathcal{L}_{1,\mu} \circ \mathcal{G}_{Q_0,\mu}) \circ (\mathcal{L}_{Q_0,\mu} \circ \mathcal{G}_{Q_0-1,\mu}) \circ \cdots \circ (\mathcal{L}_{3,\mu} \circ \mathcal{G}_{2,\mu}) \circ (\mathcal{L}_{2,\mu} \circ \mathcal{G}_{1,\mu}),$$

where the maps $\mathcal{G}_{i,\mu}$ are from (B3) and $\mathcal{L}_{i,\mu} : S_i \to S_i'$ are the local maps induced by (4.2). Our primary theorem concerns the dynamics of the family $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$.

Define $\Pi : \mathbb{S}^1 \to \mathbb{R}$ by

$$\Pi(\theta^{(1)}) = \sum_{i=1}^{Q_0} \frac{1}{\beta_{i+1}} \ln\left(\Upsilon_i(0, \theta^{(i)})\right).$$

Here $\beta_{Q_0+1} := \beta_1$. The $\theta^{(i)}$ for $2 \leqslant i \leqslant Q_0$ depend on $\theta^{(1)}$ and arise from a certain singular limit of the family $\{\mathcal{M}_\mu : 0 < \mu \leqslant \mu_0\}$. Our primary theorem assumes that Π is a Morse function.

Theorem 3.3.1. *Assume that system* (4.2) *satisfies (B1)–(B3). Suppose that the* C^3 *function* $\Pi : \mathbb{S}^1 \to \mathbb{R}$ *has finitely many nondegenerate critical points. Then there exists* $\omega_0 > 0$ *such that for any frequency* ω *satisfying* $|\omega| \geqslant \omega_0$*, there exists a set* $\Delta_\omega \subset (0, \mu_0]$ *of positive Lebesgue measure with the following property. For every* $\mu \in \Delta_\omega$*, the flow-induced map* $\mathcal{M}_\mu$ *admits a strange attractor* Ω *that supports a unique ergodic SRB measure* ν*. The orbit of Lebesgue almost every point on* S_1' *has a positive Lyapunov exponent and is asymptotically distributed according to* ν*. The SRB measure* ν *is mixing, satisfies the central limit theorem, and exhibits exponential decay of correlations for Hölder-continuous observables.*

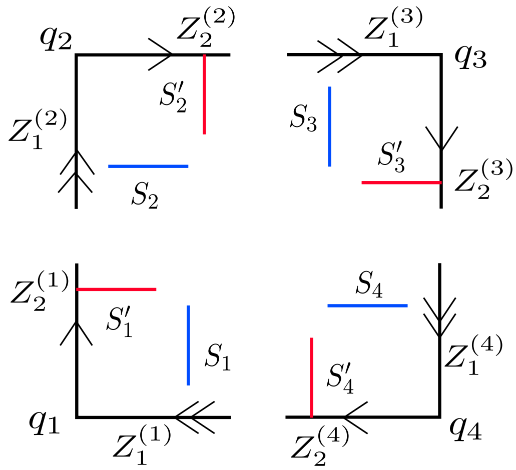

Figure 3.2: A sample configuration with 4 saddle equilibria. Pictured are the projections of the sections S_i and S_i' onto the plane.

3.3.3 Heteroclinic cycles in physical dimension at least two

Theorem 3.3.1 generalizes naturally to physical dimension $N \geqslant 2$. Here we describe the key aspects of the generalization.

First, let $-\alpha_{N-1}^{(i)} \leqslant -\alpha_{N-2}^{(i)} \leqslant \cdots \leqslant -\alpha_1^{(i)} < 0 < \beta^{(i)}$ denote the eigenvalues of $D\boldsymbol{f}(\boldsymbol{q}_i)$. We assume the following version of (B1):

(B1)* For every $1 \leqslant i \leqslant Q_0$, we have $\alpha_1^{(i)} > \beta^{(i)}$.

Second, the sections S_i and S_i' are positioned as follows. The coordinate systems $(\boldsymbol{Z}^{(i)}, \theta)$ are now given by $(\boldsymbol{Z}^{(i)}, \theta) = (Z_1^{(i)}, \ldots, Z_N^{(i)}, \theta)$ and satisfy

$$W^s(\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}) \cap (B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1) \subset \left\{(\boldsymbol{Z}^{(i)}, \theta) : Z_N^{(i)} = 0\right\}$$
$$W^u(\boldsymbol{\gamma}_{\boldsymbol{q}_i,\mu}) \cap (B(\boldsymbol{q}_i, 2\varepsilon_0) \times \mathbb{S}^1) \subset \left\{(\boldsymbol{Z}^{(i)}, \theta) : Z_1^{(i)} = \cdots = Z_{N-1}^{(i)} = 0\right\}.$$

Working in $(\boldsymbol{Z}^{(i)}, \theta)$ coordinates, for each i let H_i denote the hyperplane in $\mathbb{R}^N$ that is orthogonal to the corresponding incoming connecting orbit ($\mu = 0$) and at distance

ε_0 from saddle $\boldsymbol{q}_i$. Section S_i is positioned such that the projection of S_i onto $\mathbb{R}^N$ is a subset of H_i. Further, the projection of S_i onto the $Z_N^{(i)}$ direction is the interval $[(C_N^{(i)})^{-1}\mu, C_N^{(i)}\mu]$ for $C_N^{(i)} > 0$ suitably chosen. Section S'_i is positioned such that the projection of S'_i onto $\mathbb{R}^N$ is contained in the hyperplane that is orthogonal to the corresponding outgoing connecting orbit and at distance ε_0 from saddle $\boldsymbol{q}_i$.

Remark 3.3.1. We have formulated Theorem 3.3.1 in physical dimension two for the sake of clarity; it generalizes naturally to physical dimension $N \geqslant 2$.

3.3.4 Proof of Theorem 3.3.1

The proof of Theorem 3.3.1 closely follows the proof of Theorem 3.2.2 given in Section 3.2.4. We therefore present only the modifications of the argument given in Section 3.2.4 that are needed for the heteroclinic setting.

In magnified coordinates, for each $1 \leqslant i \leqslant Q_0$ the global map $\mathcal{G}_{i,\mu}$ is given by

$$(z_1^{(i)}, z_2^{(i)} = \varepsilon_0\mu^{-1}, \theta) \mapsto (y_1, y_2, \gamma),$$

where

$$y_1 = \varepsilon_0\mu^{-1} \tag{3.18a}$$

$$y_2 = b_i z_1^{(i)} + \Upsilon_i(\mu z_1^{(i)}, \theta) \tag{3.18b}$$

$$\gamma = \theta + \zeta_i + \mu\Psi_i(\mu z_1^{(i)}, \theta). \tag{3.18c}$$

The local map $\mathcal{L}_{i,\mu}$ is given by

$$(z_1^{(i)}, z_2^{(i)}, \theta) \mapsto (x_1, x_2, \rho),$$

where

$$x_1 = \varepsilon_0\mu^{-1}\left(\frac{\varepsilon_0\mu^{-1}}{z_2^{(i)}}\right)^{\frac{-\lambda_i+w_1^{(i)}}{\beta_i+w_2^{(i)}}} \tag{3.19a}$$

$$x_2 = \varepsilon_0\mu^{-1} \tag{3.19b}$$

$$\rho = \theta + \left(\frac{\omega}{\beta_i + w_2^{(i)}}\right)\ln\left(\frac{\varepsilon_0\mu^{-1}}{z_2^{(i)}}\right). \tag{3.19c}$$

Here $w_1^{(i)}$ and $w_2^{(i)}$ are defined as in (3.9).

We use (3.18c) and (3.19c) to compute the angular component of the flow-induced map

$$\mathcal{M}_\mu = (\mathcal{L}_{1,\mu}\circ\mathcal{G}_{Q_0,\mu})\circ(\mathcal{L}_{Q_0,\mu}\circ\mathcal{G}_{Q_0-1,\mu})\circ\cdots\circ(\mathcal{L}_{2,\mu}\circ\mathcal{G}_{1,\mu}).$$

Let $(x_1^{(1)}, x_2^{(1)} = \varepsilon_0\mu^{-1}, \theta^{(1)}) \in S_1'$. For $1 \leqslant i \leqslant Q_0$, define

$$(x_1^{(i)}, x_2^{(i)} = \varepsilon_0\mu^{-1}, \theta^{(i)}) = (\mathcal{L}_{i,\mu}\circ\mathcal{G}_{i-1,\mu})\circ\cdots\circ(\mathcal{L}_{2,\mu}\circ\mathcal{G}_{1,\mu})\,(x_1^{(1)}, x_2^{(1)}, \theta^{(1)}).$$

The flow-induced map $\mathcal{M}_\mu$ is given by $(x_1^{(1)}, x_2^{(1)} = \varepsilon_0\mu^{-1}, \theta^{(1)}) \mapsto (z_1, z_2 = \varepsilon_0\mu^{-1}, \hat{\theta})$, where $\hat{\theta}$ is computed using (3.18c) and (3.19c):

$$\hat{\theta} = \theta^{(1)} + \sum_{i=1}^{Q_0}\zeta_i + \mu\Psi_i(\mu x_1^{(i)}, \theta^{(i)}) + \left(\frac{\omega}{\beta_{i+1} + w_2^{(i+1)}}\right)\ln\left(\frac{\varepsilon_0\mu^{-1}}{b_i x_1^{(i)} + \Upsilon_i(\mu x_1^{(i)}, \theta^{(i)})}\right). \tag{3.20}$$

As in the homoclinic case, we compute the singular limit of $\left\{\mathcal{M}_{\mu(p)} : p \in [\ln(\mu_0^{-1}), \infty)\right\}$ by deriving an auxiliary parameter a from p. Define $\kappa : (0,\infty) \to \mathbb{R}$ by

$$\kappa(s) = \sum_{i=1}^{Q_0}\frac{\omega}{\beta_{i+1}}\ln(s^{-1}).$$

Let $(\mu_n)_{n=1}^{\infty}$ be any strictly decreasing sequence such that $\mu_n \in (0, \mu_0]$ for all $n \in \mathbb{N}$, $\mu_n \to 0$ as $n \to \infty$, and $\kappa(\mu_n) \in 2\pi\mathbb{Z}$ for all $n \in \mathbb{N}$. For $a \in \mathbb{S}^1$, define

$$\mu_{a,n} = \kappa^{-1}(\kappa(\mu_n) + a), \qquad p(a,n) = \ln(\mu_{a,n}^{-1}).$$

We view the family of flow-induced maps as a two-parameter family of embeddings: $\left\{\mathcal{M}_{\mu(p(a,n))} : a \in \mathbb{S}^1,\ \ n \in \mathbb{N}\right\}$. The following proposition establishes C^3 convergence to a singular limit as $n \to \infty$.

Proposition 3.3.2. *We have*

$$\lim_{n\to\infty} \left\|\mathcal{M}_{\mu(p(a,n))} - (0, \mathcal{F}_a)\right\|_{C^3(S_1' \times [0,2\pi))} = 0,$$

where $\mathcal{F}_a : S_1' \to \mathbb{S}^1$ *is given by*

$$\mathcal{F}_a(x_1^{(1)}, \theta^{(1)}) = \theta^{(1)} + a + \left(\sum_{i=1}^{Q_0} \zeta_i + \frac{\omega}{\beta_{i+1}} \ln(\varepsilon_0)\right) - \sum_{i=1}^{Q_0} \frac{\omega}{\beta_{i+1}} \ln\left(b_i x_1^{(i)} + \Upsilon_i(0, \theta^{(i)})\right). \tag{3.21}$$

Proof of Proposition 3.3.2. The proof of Proposition 3.3.2 uses (3.20) and follows the line of reasoning developed in the proof of Proposition 3.2.5. ■

We finish the proof of Theorem 3.3.1 by showing that the family of flow-induced maps $\left\{\mathcal{M}_{\mu(p(a,n))} : a \in \mathbb{S}^1,\ \ n \in \mathbb{N}\right\}$ satisfies (H1)–(H7). The distortion bound (H1)(c) follows from the fact that the distortion of each local and global map is bounded. Hypotheses (H2) and (H3) follow from Proposition 3.3.2. Hypotheses (H4), (H5), and (H7) concern the family of circle maps $\{h_a : \mathbb{S}^1 \to \mathbb{S}^1,\ \ a \in \mathbb{S}^1\}$ defined by setting $x_1^{(1)} = 0$ in (3.21):

$$h_a(\theta^{(1)}) := \mathcal{F}_a(0, \theta^{(1)}) = \theta^{(1)} + a + \left(\sum_{i=1}^{Q_0} \zeta_i + \frac{\omega}{\beta_{i+1}} \ln(\varepsilon_0)\right) - \sum_{i=1}^{Q_0} \frac{\omega}{\beta_{i+1}} \ln\left(\Upsilon_i(0, \theta^{(i)})\right).$$

Since

$$\sum_{i=1}^{Q_0} \frac{1}{\beta_{i+1}} \ln\left(\Upsilon_i(0, \theta^{(i)})\right)$$

is a Morse function by hypothesis, (H4), (H5), and (H7) follow from Proposition (3.1.2) if $|\omega|$ is sufficiently large. Finally, the nondegeneracy condition (H6) follows by direct computation using (3.21) and the fact that $b_1 \neq 0$.

3.3.5 An asymmetric May-Leonard model

Theorem 3.3.1 and its generalization to any finite physical dimension applies to numerous concrete systems of interest in the biological and physical sciences. We mention one such system here. The asymmetric May-Leonard flow is the flow on the nonnegative octant of $\mathbb{R}^3$ generated by

$$\begin{aligned}
\dot{x}_1 &= x_1(1 - x_1 - a_1 x_2 - b_1 x_3) \\
\dot{x}_2 &= x_2(1 - b_2 x_1 - x_2 - a_2 x_3) \\
\dot{x}_3 &= x_3(1 - a_3 x_1 - b_3 x_2 - x_3).
\end{aligned} \tag{3.22}$$

System (3.22) models the Lotka-Volterra dynamics of three competing species with

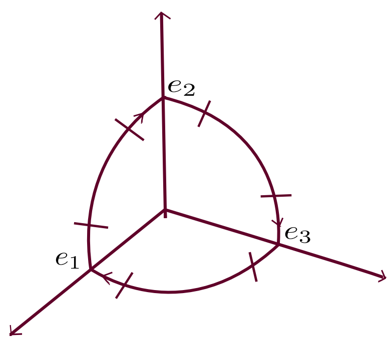

Figure 3.3: heteroclinic cycle with cross sections.

equal intrinsic growth rates and differing competition coefficients. Assuming $0 < a_i < 1 < b_i < 2$ for $1 \leqslant i \leqslant 3$, (3.22) admits a heteroclinic cycle with saddles $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ (see Figure 3.3).The asymptotic stability of this cycle is studied in [22]. If the competition coefficients also satisfy

$$\frac{b_1 - 1}{1 - a_2} > 1, \qquad \frac{b_2 - 1}{1 - a_3} > 1, \qquad \frac{b_3 - 1}{1 - a_1} > 1,$$

then (B1)* is satisfied and therefore the generalization of Theorem 3.3.1 to physical dimension three applies to the periodically-forced May-Leonard system with any C^4 periodic forcing functions $p_i \quad (1 \leq i \leq 3)$, given by

$$\begin{cases} \dot{x}_1 = x_1(1 - x_1 - a_1x_2 - b_1x_3) + \mu p_1(\boldsymbol{x}, \omega t) \\ \dot{x}_2 = x_2(1 - b_2x_1 - x_2 - a_2x_3) + \mu p_2(\boldsymbol{x}, \omega t) \\ \dot{x}_3 = x_3(1 - a_3x_1 - b_3x_2 - x_3) + \mu p_3(\boldsymbol{x}, \omega t). \end{cases}$$

CHAPTER 4

Normal Form and Error Estimates

In this Chapter we prove Propositions 3.2.1, 3.2.4, and 3.2.3. Description of properties of trajectories staying for a long time near saddle fixed points can be easily done when a system is reduce to a certain linear form. Proposition 3.2.1 is about reduction of a periodically forced system to a normal form near saddle point. We use this Proposition to derive the local map by integrating the systems near saddle. Proposition 3.2.4 gives C^3 control over time spend by the solutions near saddle and Proposition 3.2.3 control the nonlinear part of the solutions in C^3 norm.

4.1 Normal form

Proof of Proposition 3.2.1:

We begin with the equation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{\xi}) \tag{4.1}$$

and its perturbed form as discussed in Chapter 3

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{\xi}) + \mu\boldsymbol{p}(\boldsymbol{\xi}, \theta) \tag{4.2a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega. \tag{4.2b}$$

By hypothesis (A3) on local dynamics from section (3.2) of Chapter 3, we have an analytic coordinate change

$$\boldsymbol{\eta} = \boldsymbol{\xi} + \boldsymbol{q} \text{ with inverse } \boldsymbol{\xi} = \boldsymbol{\eta} + \boldsymbol{Q}$$

where $q_i, Q_i : \mathbb{R}^N \to \mathbb{R} \quad (1 \leq i \leq N)$ are analytic, such that equation (4.1) takes the following linear form

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}t} = D\boldsymbol{f}(\boldsymbol{0})\boldsymbol{\eta}.$$

Let's denote the eigen value with respect to unstable direction $\beta = -\alpha_N > 0$ through out this proof. Using this coordinate change and equation (4.1) and (4.2a), we get

$$(1 + \partial x_i Q_i)(-\alpha_i x + f_i) + \sum_{j \neq i} (\partial x_j Q_j)(\alpha_j x_j + f_j) = -\alpha_i \eta_i \quad (1 \leq i, j \leq N) \tag{4.3}$$

$$\frac{d\eta_i}{dt} = -\alpha_i \eta_i + \mu(1 + \partial x_i Q_i - \sum_{j \neq i} \partial x_j Q_j)(\mu p_i) \quad (1 \leq i, j \leq N) \tag{4.4}$$

Using equation (4.4) and (4.2), we get the following form of equation (4.2)

$$\frac{\mathrm{d}\eta_i}{\mathrm{d}t} = -\alpha_i \eta_i + \mu(1 + h_i(\boldsymbol{\eta}))(\hat{p}_i(\boldsymbol{\eta}, \boldsymbol{\theta})) \quad (1 \le i \le N) \tag{4.5a}$$

$$\theta^{'} = w \tag{4.5b}$$

where $h_i(\boldsymbol{\eta}) = 1 + \partial x_i Q_i(\boldsymbol{x}) - \sum_{j \neq i} \partial x_j Q_j(\boldsymbol{x})$ and $\hat{p}(\boldsymbol{\eta}, \theta) = p(\boldsymbol{x}, \theta)$. When force is added to the unperturbed system the hyperbolic stationary point of the equation (4.1) became hyperbolic periodic solution of the equation (4.2) with period $2\pi w^{-1}$. To standardize the location of periodic orbit and its local stable and unstable manifolds we use the following steps.

Standardization of periodic orbit

let $\mu\boldsymbol{\phi}(\theta, \mu) = \mu\boldsymbol{\phi}(\theta + 2\pi, \mu)$ be periodic solutions of the equation (4.5a). We proceed to show that such solutions exist and unique with bounded C^3 norm. The functions ϕ_i for $1 \le i \le N$ should satisfy

$$w\frac{\mathrm{d}\phi_i}{d\theta} = -\alpha_i \phi_i - (1 + h_i(\mu\boldsymbol{\phi}))(\hat{p}_i(\mu\boldsymbol{\phi}, \theta)) \tag{4.6}$$

So we have

$$\phi_i(\theta, \mu) = e^{-\alpha_i w^{-1}(\theta - \theta_0)} - w^{-1} \int_{\theta_0}^{\theta} e^{\alpha_i w^{-1}(s - \theta_0)} [1 + h_i(\mu\boldsymbol{\phi}(s, \mu))] \cdot [\hat{p}_i(\mu\boldsymbol{\phi}(s, \mu))] \mathrm{d}s$$

Let $\theta = \theta_0 + 2\pi$. Using $\phi_i(\theta_0 + 2\pi; \mu) = \phi_i(\theta_0, \mu)$ for $1 \le i \le N$, we get the following

$$\phi_i(\theta; \mu) = \frac{-w^{-1}}{1 - e^{-2\alpha_i w^{-1}\pi}} \int_0^{2\pi} e^{\alpha_i w^{-1}(s - 2\pi)} [1 + h_i(\mu\boldsymbol{\phi}(s + \theta); \mu)] \cdot [\hat{p}_i(\mu\boldsymbol{\phi}(s + \theta); \mu)] \mathrm{d}s$$

The existence and uniqueness of the functions ϕ_i follows from contraction mapping theorem. The bound on the partial derivatives with respect to θ and μ obtained from above equation by taking the partial derivatives. Now introducing a new coordinate change

$$\boldsymbol{X} = \boldsymbol{\xi} - \mu\boldsymbol{\phi}(\theta;\mu) \tag{4.7}$$

which stabilizes the periodic orbit to a saddle point. In terms of this coordinates, equation (4.5) has the following form

$$\begin{aligned}\frac{\mathrm{d}X_i}{dt} &= -\alpha_i X_i + \mu F_i(\boldsymbol{X},\theta;\mu) \quad (1 \le i \le N)\\ \frac{\mathrm{d}\theta}{dt} &= w\end{aligned} \tag{4.8}$$

where for each i

$$F_i(\boldsymbol{X},\theta;\mu) = -[h_i(\boldsymbol{X}+\mu\boldsymbol{\phi}) - h_i(\mu\boldsymbol{\phi})](\hat{p}_i(\boldsymbol{X}+\mu\boldsymbol{\phi})) - (1+h_i(\mu\boldsymbol{\phi})(\hat{p}_i(\boldsymbol{X}+\mu\boldsymbol{\phi}) - \hat{p}_i(\mu\boldsymbol{\phi}))$$

and are analytic with bounded C^3 norm on

$$\{(\boldsymbol{X},\theta;\mu) : \|\boldsymbol{X}\| < \varepsilon, \theta \in \mathbb{S}^1, 0 \le \mu \le \mu_0\}$$

Flattening local manifolds

Let

$$X_n = \mu W^s(X_1 \cdots X_{N-1}, \theta; \mu)$$

$$X_i = \mu W_i^u(X_n, \theta; \mu) \quad (1 \le i \le N)$$

be local stable and unstable manifolds of the periodic solution $(\boldsymbol{X},\theta) = (\boldsymbol{0}, wt)$ of equation (4.8). We use following standard result about local manifolds.

Proposition 4.1.1. *There exist $\varepsilon > 0$ and $\mu_0 = \mu_0(\varepsilon) > 0$ such that W_i^u and W^s are analytically defined on*

$$(-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times [0, \mu_0]$$

and satisfy

$$W_i^u(0,\theta;\mu) = 0 \ \ (1 \le i \le N) \ \ \textit{and} \quad W^s(0,\theta;\mu) = 0$$

with

$$\|W_i^u\|_3 \leq K \ \ (1 \leq i \leq N) \ \ and \ \ \|W^s\|_3 \leq K$$

Proof. See [31] for the proof. ■

By definition W_i^u for $1 \leq i \leq N-1$ and W^u satisfies

$$-\alpha_i W_i^u + F_i = w\partial_\theta W_i^u + \partial_{X_N} W_j^u \cdot -\alpha_N X_N + \mu F_N \tag{4.9a}$$

$$-\alpha_n W^s + F_n = w\partial_\theta W^s + \sum_{i=1}^{N-1}(\partial_{X_i} W^s \cdot -\alpha_i X_i) + \mu F_i \tag{4.9b}$$

Now define new coordinates

$$x_i = X_i - W_i^u \ \ (1 \leq i \leq N-1)$$

$$x_n = X_n - W^s$$

By using the equation (4.9) and new coordinates systems defined above (4.8) takes the following form

$$\begin{aligned} \frac{dx_i}{dt} &= (-\alpha_i x_i + \mu G_i(\boldsymbol{x}, \theta; \mu))x_i \quad (1 \leq i \leq N) \\ \frac{d\theta}{dt} &= w \end{aligned} \tag{4.10}$$

where for each G_i is a analytic function of all arguments defined on $U_\varepsilon \times \mathbb{S}^1 \times [0, \mu_0]$ with C^3 norm bounded by constant K.

This completes the proof of the Proposition 3.2.1.

4.2 Error estimates

4.2.1 Proof of Proposition 3.2.3

Let $z = \prod_{i=1}^{i=n} x_i^{d_i}.\mu^{d_{n+1}}$ and let ∂_{z^k} denote the corresponding partial derivatives with order $k = \sum_{i=1}^{n+1} d_i$. Since the C^3 norms of the functions G_i are bounded on $\mathcal{U}_\varepsilon \times [0, \mu_0]$ and $G_i(\boldsymbol{x}) = g_i(\mu \boldsymbol{x})$, there exist $K > 0$ such that for every z of order $\leq n$, we have

$$|\partial_z^k(\partial_{\theta_i}^i g_i.z)| \leq K \qquad (1 \leq i \leq n) \tag{4.11}$$

on the domain $\mathcal{D}(t, q_0, p)$.

C^0 estimates

Using the inequality (4.11) and Proposition 3.4, we have

$$\|w_i\|_{C^0(\mathcal{D}(t,q_0,p))} \leq K\mu \qquad (1 \leq i \leq N) \tag{4.12}$$

C^1 estimates

$\theta_t = \theta_0 + wt$ so $\partial_{x_{i,0}}\theta = 0$. Now using (3.9), we have

$$\partial_{x_{i,0}} w_j = \mu t^{-1} \int_0^t (\sum_k \partial_{x_k} g_j.\partial_{x_{i,0}} x_k) \mathrm{d}s \qquad (1 \leq i, j \leq N) \tag{4.13}$$

and from equation (3.8), we have following for $1 \leq i, j \leq N$

$$\partial_{x_{i,0}} x_j = t x_j \partial_{x_{i,0}} w_j \qquad (j \neq i) \tag{4.14}$$

$$\partial_{x_{i,0}} x_i = t x_i \partial_{x_{i,0}} w_i + \frac{x_i}{x_{i,0}} \tag{4.15}$$

using equations (4.13), (4.14), and, (4.15), we get

$$\partial_{x_{i,0}} w_j = \mu t^{-1} \int_0^t (\sum_k \partial_{x_k} g_j.x_k.s\partial_{x_{i,0}} w_j) \mathrm{d}s + \mu t^{-1} \int_0^t \partial_{x_i} g_i.\frac{x_i}{x_{i,0}} \mathrm{d}s \quad (1 \leq i, j \leq N) \tag{4.16}$$

from equation (4.11), we know that

$$|\partial_{x_i} g_j . x_i| < K \qquad (1 \leq i, j \leq N) \tag{4.17}$$

using the above inquality in (4.16), we get the following

$$|\partial_{x_{i,0}} w_j| \leq K\mu t^{-1} \int_0^t (\sum_j |s\partial_{x_{i,0}} w_j|) ds + \widehat{K}\mu \qquad (1 \leq i, j \leq N) \tag{4.18}$$

from which it follows that

$$|\partial_{x_{i,0}} w_j| < K\mu. \qquad (1 \leq i \leq N)$$

And for $\partial_{\theta_0} w_i$ $(1 \leq i \leq N)$ we use $\partial_{\theta_0} \theta = 1$. Computing the derivatives

$$\partial_{\theta_0} w_i = \mu t^{-1} \int_0^t \sum_j (\partial_{x_j} g_i . \partial_{\theta_0} x_j) + \partial_{\theta_0} g_i \mathrm{d}s \tag{4.19}$$

using

$$\partial_{\theta_0} x_j = t x_j \partial_{\theta_0} w_j \qquad (1 \leq i \leq N)$$

in equation (4.19), we conclude

$$|\partial_{\theta_0} w_i| < K\mu. \qquad (1 \leq i \leq N)$$

For $\partial_p w_j$ we use the facts $\partial_p \mu = \mu, \partial_p g_j = \mu \partial_\mu g_j$. By using the similar arguments as above it follows that $\partial_p w_i$ for $1 \leq i \leq N$ are bounded by K. Boundedness of $\partial_t w_i$ for $1 \leq i \leq N$ follows from C^0 estimates. Hence C^1 estimates follows.

C^2 estimates

We use equation (4.13) to compute the 2nd derivatives.

$$\partial^2_{x_{i,0} x_{i,0}} w_j = \mu t^{-1} \int_0^t (\sum_k (\sum_l \partial^2_{x_k x_l} g_j . \partial_{x_{i,0}} x_l) \partial_{x_{i,0}} x_k + \sum_k (\partial_{x_k} g_j . \partial^2_{x_{i,0} x_{i,0}} x_k)) \mathrm{d}s \qquad (1 \leq i, j \leq N) \tag{4.20}$$

using (4.14), we have

$$\partial^2_{x_{i,0}x_{i,0}} x_j = t\partial_{x_{i,0}} x_j \partial_{x_{i,0}} w_j + t x_j \partial^2_{x_{i,0}x_{i,0}} w_j \qquad (j \neq i) \tag{4.21}$$

$$\partial^2_{x_{i,0}x_{i,0}} x_i = t\partial_{x_{i,0}} x_i \partial_{x_{i,0}} w_i + t x_i \partial^2_{x_{i,0}x_{i,0}} w_i + \frac{\partial_{x_{i,0}} x_i}{x_{i,0}} - \frac{x_i}{x_{i0}^2} \tag{4.22}$$

for $1 \leq i, j \leq N$

using equation (4.21), and (4.22) in (4.20), we get

$$\begin{aligned}\partial^2_{x_{i,0}x_{i,0}} w_j = \mu t^{-1} &\int_0^t \sum_k (\sum_l \partial^2_{x_k x_l} g_j . \partial_{x_{i,0}} x_l) \partial_{x_{i,0}} x_k) \mathrm{d}s \\ &+ \mu t^{-1} \int \sum_k (\partial_{x_k} g_j . t\partial_{x_{i,0}} x_k \partial_{x_{i,0}} w_k + t x_j \partial^2_{x_{i,0}x_{i,0}} w_k) \mathrm{d}s \\ &+ \mu t^{-1} \int \partial_{x_k} g_j . (\frac{\partial_{x_{i,0}} x_i}{x_{i,0}} - \frac{x_i}{x_{i0}^2}) \mathrm{d}s\end{aligned} \tag{4.23}$$

for $1 \leq i, j \leq N$

using equation (4.14), (4.11), and, C^1 estimates on $w_i, 1 \leq i \leq N$ in equation (4.23), we conclude that

$$|\partial^2_{x_{i,0}x_{i,0}} w_j| < K\mu. \qquad (1 \leq i, j \leq N)$$

Bounds for the other second derivatives are shown using similar procedure. And C^3 estimates are also found in same spirit. This completes the proof.

4.2.2 Proof of Proposition 3.2.4

From Chapter 3 equation (3.2.5) we have stopping time

$$T(\boldsymbol{q}_0, p) = \frac{1}{\beta + w_N(T(\boldsymbol{q}_0, p), \boldsymbol{q}_0; \mu)} \ln \left(\frac{\varepsilon_0 \mu^{-1}}{x_{N,0}} \right) = \frac{1}{\beta + w_N} \ln \left(\frac{\varepsilon_0 \mu^{-1}}{x_{N,0}} \right) \tag{4.24}$$

Let $T(\boldsymbol{q}_0, p) = T$. C^0 estimates of $T - \frac{1}{\beta}\ln(\varepsilon_0\mu^{-1})$ on the specified domain follows from the fact that $|w_N| < K\mu$ and $x_{N,0} \in \Gamma^2$.

C^1 estimates

From equation (4.24), we have

$$\partial_{x_{i,0}} T = log(\frac{\varepsilon\mu^{-1}}{x_{N,0}}).\frac{\partial_{x_{i,0}} w_N}{(\alpha_n + w_N)^2} \quad (1 \leq i \leq N-1) \tag{4.25}$$

and

$$\partial_{x_{N,0}} T = log(\frac{\varepsilon\mu^{-1}}{x_{N,0}}).\frac{\partial_{x_{N,0}} w_N}{(\alpha_n + w_N)^2} + \frac{1}{(\alpha_n + w_N)x_{N,0}} \tag{4.26}$$

using C^0 estimates of T and C^1 estimates of w_N we have

$$|\partial_{x_{i,0}} T| < K_5. \quad (1 \leq i \leq N) \tag{4.27}$$

Similarly

$$|\partial_\theta T| < K_5.$$

For μ derivative we use $\partial_p \mu = \mu$

$$\partial_p T = log(\frac{\varepsilon\mu^{-1}}{x_{N,0}})\frac{\partial_p w_N \cdot \mu}{(\alpha_n + w_N)^2} + \frac{1}{\alpha_N + w_N} \tag{4.28}$$

by same reasons

$$|\partial_p T| < k_5.$$

C^1 estimates follows. C^2 and C^3 estimates are follows from computing the derivatives and using C^1 and C^0 estimates of w_N and T. This completes the proof of Proposition 3.2.4.

CHAPTER 5

Statistical Properties of Dynamical Systems

We study memory loss in nonequilibrium open dynamical systems. This work fits into the larger study of statistical properties of dynamical systems. Memory loss in this setting is an analog of decay of correlations. Transfer operators play a central role in memory loss results.

5.1 Transfer operator

Let (X, T) be a smooth dynamical system on a compact manifold X. The map T associates to each measurable $\varphi : X \to \mathbb{R}$ the function $P_T(\varphi) : X \to \mathbb{R}$ defined by

$$(\mathcal{P}_T\varphi)(x) = \sum_{y:T(y)=x} \frac{\varphi(y)}{|T'(y)|}$$

By change of variables it follows that for $\varphi \in \mathbb{L}^\infty$ and $\psi \in \mathbb{L}^1$, we have

$$\int (\varphi \circ T) \cdot \psi \ \mathrm{d}x = \int \varphi \cdot (\mathcal{P}_T \psi) \ \mathrm{d}x$$

If φ is the density of an absolutely continuous measure μ, then $\mathcal{P}_T(\varphi)$ is the density of the push-forward measure $T_*\mu$ defined by $T_*\mu(A) = \mu(T^{-1}(A))$. We can view $\mathcal{P}_T$ as the action induced by T on the space $\mathcal{M}$ of absolutely continuous measures. It has the following nice properties.

(a) $\mathcal{P}_T$ is linear and positive.

(b) $\|P_T(\varphi)\|_1 \leq \|\varphi\|_1 \ \forall \varphi \in \mathbb{L}^1$.

(c) $\mathcal{P}_T \varphi = \varphi$ iff $\mu = \varphi \ \mathrm{d}x$ is an absolutely continuous T-invariant measure.

(d) $\mathcal{P}_{T^k} = \mathcal{P}_T^k \ \forall$ positive integer k.

The map $\mathcal{P}_T$ is called the transfer operator; It describes the evolution of initial densities under the dynamics. The transfer operator is used to find absolutely continuous invariant measures and study statistical properties such as decay of correlations.

5.2 Statistical methods

One can use the following ideas when studying statistical properties of dynamical systems.

(a) Spectral properties of the transfer operator and tools from functional analysis.

(b) Coupling techniques from probability theory. This involves matching and evolution of densities under the transfer operator.

(c) Hilbert or projective metric method. This uses a contraction theorem for the Hilbert metric and gives explicit rates of correlation decay.

We use the Hilbert or projective metric method to study memory loss for nonequilibrium open dynamical systems.

5.2.1 Convex cones and Hilbert metric

We start with definition of convex cone.

Definition 5.2.1. Let $\mathcal{V}$ be a vector space. A *convex cone* is a subset $\mathcal{C} \subset \mathcal{V}$ with the following properties.

(a) $\mathcal{C} \cap -\mathcal{C} = \emptyset$

(b) $\gamma\mathcal{C} = \mathcal{C}$ for all $\gamma > 0$

(c) $\mathcal{C}$ is a convex set

(d) For all $\varphi, \psi \in \mathcal{C}$, every $c \in \mathbb{R}$, and every sequence (c_n) in $\mathbb{R}$ such that $c_n \to c$, if $\varphi - c_n\psi \in \mathcal{C}$ for all n, then $\varphi - c\psi \in \mathcal{C} \cup \{0\}$.

Example 5.2.2. Let $\mathcal{V} = BV([0,1], \mathbb{R})$ be the space of all real valued functions of bounded variation on the unit interval. Let $\mathcal{C} = \{\varphi \in \mathcal{V} : \varphi \geq 0, \varphi \not\equiv 0\}$ and $\mathcal{C}_a = \{\varphi \in \mathcal{V} : \phi \geq 0, \varphi \not\equiv 0, \operatorname{var}(\varphi) \leq a \int \varphi d\mu\}$ for $a > 0$. Then $\mathcal{C}$ and $\mathcal{C}_a$ are convex cones.

Definition 5.2.3. Let $\mathcal{C}$ be a convex cone and let $\phi, \psi \in \mathcal{C}$. Let $\alpha(\varphi, \psi) = \inf\{c > 0 : c\varphi - \psi \in \mathcal{C}\}$ and $\beta(\varphi, \psi) = \sup\{r > 0 : \psi - r\varphi \in \mathcal{C}\}$. The Hilbert metric

$d_{\mathcal{C}}$ is defined on $\mathcal{C}$ by

$$d_{\mathcal{C}}(\varphi, \psi) = \log \left(\frac{\alpha(\varphi, \psi)}{\beta(\varphi, \psi)} \right).$$

Where we take $\alpha = \infty$ and $\beta = 0$ if the corresponding sets are empty.

Observe that $d_{\mathcal{C}}$ is projective, i.e $d_{\mathcal{C}}(\lambda_1 \varphi, \lambda_2 \psi) = d_{\mathcal{C}}(\varphi, \psi)$ for λ_1, λ_2 positive. The following result due to G.Birkhoff asserts that in the current context, a positive linear operator is a contraction in the Hilbert metric provided the diameter of the image is finite.

Theorem 5.2.4 ([12]). *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be vector spaces containing convex cones $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Let $\mathcal{L} : \mathcal{V}_1 \to \mathcal{V}_2$ be a positive linear operator, meaning $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$. Define*

$$\Delta = \sup_{\varphi^*, \psi^* \in \mathcal{L}(\mathcal{C}_1)} d_{\mathcal{C}_2}(\varphi^*, \psi^*).$$

Then for all $\varphi, \psi \in \mathcal{C}_1$, we have

$$d_{\mathcal{C}_2}(\mathcal{L}\varphi, \mathcal{L}\psi) \leqslant \tanh\left(\frac{\Delta}{4}\right) d_{\mathcal{C}_1}(\varphi, \psi).$$

This result allows one to obtain explicit, constructive bounds on rates of correlation decay. It is natural to relate the Hilbert metric to some familiar norm. The following proposition due to Liverani, Saussol and Vaienti describes the relation of Hilbert metric to adapted norms on $\mathcal{V}$.

Proposition 5.2.5. *[53] Let $\mathcal{C} \subset \mathcal{V}$ be a convex cone and let $\|\cdot\|$ be an adapted norm on $\mathcal{V}$; that is, a norm such that for all $\varphi, \psi \in \mathcal{V}$, if $\psi - \varphi \in \mathcal{C}$ and $\psi + \varphi \in \mathcal{C}$, then $\|\varphi\| \leqslant \|\psi\|$. Then for all $\varphi, \psi \in \mathcal{C}$, we have*

$$\|\varphi\| = \|\psi\| \Longrightarrow \|\varphi - \psi\| \leqslant \left(e^{d_{\mathcal{C}}(\varphi, \psi)} - 1\right) \|\varphi\|.$$

Remark 5.2.1. $\|.\|_1$ is adapted with respect to the cone $\mathcal{C}$ of all nonnegative, bounded variation functions in $\mathbb{L}^1[0,1]$.

The idea is to consider the cones defined in the example 5.2.2 as sets of density functions on which transfer operators will act. We want to show that such actions are contractions in the Hilbert metric and thereby develop results on memory loss.

5.3 Memory Loss

Memory is lost if the state of the system is largely independent of the initial state as time increases. There are two ways memory loss can happen conceptually. First trajectories may coalesce near a single trajectory as they evolve. The following example illustrates the idea.

Example 5.3.1. Let X be a compact metric space and $f_i : X \to X$ be a sequence of uniformly Lipschitz maps with Lipschitz constant $L < 1$. The dynamical system is defined by composition these maps. Since each map is contracting, all trajectories coalesce into a small blob which continues to evolve with time.

A similar phenomenon occurs in random dynamical systems. An SDE of the form

$$\mathrm{d}x_t = a(x_t)\,\mathrm{d}t + \sum_{i=1}^{n} b_i(x_t) \circ \mathrm{d}W_t^i$$

gives rise to a stochastic flow of diffeomorphisms in which almost every Brownian path defines a time-dependent flow [42]. When all Lyapunov exponents are strictly negative, trajectories coalesce near a unique equilibrium point that evolves in time called a random sink [44]. This phenomenon occurs in dissipative systems such as the

Navier-Stokes system [57, 58] and in certain coupled oscillator networks modelling neuronal activity [49].

In chaotic system memory is lost because of sensitive dependence on initial conditions. Small errors in initial conditions can lead to substantial errors, so in practice it is impossible to track specific trajectories in such systems. A statistical approach is often considered. We say that an autonomous system exhibits *memory loss* in the statistical sense if there exists a unique invariant measure with density φ such that for any suitable absolutely continuous initial distribution with density ψ_0, we have $\psi_t \to \varphi$ as $t \to \infty$, where ψ_t is the dynamical evolution of ψ_0. We say a non-equilibrium system **loses memory in the statistical sense** if for any two suitable initial densities φ_0 and ψ_0, we have

$$\int |\varphi_t - \psi_t| \mathrm{d}\mu \to 0 \ \ as \ \ t \to \infty$$

where μ is a reference measure. In the following sections we discuss some existing memory loss results on time-dependent systems.

5.3.1 Expanding maps

Definition 5.3.2. Let M be a compact connected Riemannian manifold with Riemannian volume m. A smooth map $f : M \to M$ is said to be **expanding** if there exists $\lambda > 1$ such that $|D_x f(v)| \geq \lambda |v|$ for every $x \in M$ and every tangent vector v at x.

Let $f_i : M \to M$ be a sequence of expanding maps. The dynamical system is defined by the composition of expanding maps. The map at time m, F_m, is defined by $F_m = f_m \circ \cdots \circ f_1$. For $\lambda > 0$ and $\Gamma \geq 0$ define

$$\mathcal{E}(\lambda, \Gamma) = \{f : M \to M : \|f\|_{C^2} \leq \Gamma, |Df(x)v| \geq \lambda|v| \ \ \forall (x,v) \in TM\}$$

and let the class of density functions be given by

$$\mathcal{D} := \{\varphi > 0 : \int \varphi \mathrm{d}m = 1, \varphi \ \ is \ \ \text{Lipschitz}\}$$

Theorem 5.3.3. *[68] Given $\lambda > 1$ and $\Gamma > 0$, there exist a constant $\Lambda = \Lambda(\lambda, \Gamma) \in (0,1)$ such that for any sequence $f_i \in \mathcal{E}(\lambda, \Gamma)$ and any $\varphi, \psi \in \mathcal{D}$, there exists $C_{\varphi,\psi}$ such that*

$$\int |\mathcal{P}_{F_n}(\varphi) - \mathcal{P}_{F_n}(\psi)| \mathrm{d}m \leq C_{\varphi,\psi} \Lambda^n \ \ \forall n \geq 0.$$

The proof uses a coupling technique.

5.3.2 Piecewise expanding maps in dimension one

Definition 5.3.4. Let $\mathcal{S}^1$ be the interval $[0,1]$ with end points identified. A map $f : \mathcal{S}^1 \to \mathcal{S}^1$ is said to be piecewise C^2 expanding if there exists a finite partition $\mathcal{Q}$ of $\mathcal{S}^1$ into intervals such that for each $J \in \mathcal{Q}$

(a) $f|J$ extends to a C^2 mapping in a neighborhood of J

(b) there exist $\lambda > 1$ such that $|f^{'}(x)| \geq \lambda$ for all $x \in J$.

Iterates of a single piecewise C^2 expanding map may not exhibit memory loss. Indeed memory loss in this context is equivalent to mixing, and a single piecewise C^2 expanding map may need not even be ergodic. For this reason we introduce enveloping maps.

Definition 5.3.5. Let f be a piecewise C^2 expanding map. For $n \in \mathbb{N}$ define $\mathcal{Q}_n := \bigvee_{i=1}^{n} f^{-(i-1)}\mathcal{Q}$. We say f is enveloping if there is $N \in \mathbb{Z}^+$ such that for every $J \in \mathcal{Q}$ we have $\bigcup_{I\in\mathcal{Q}_{\mathcal{N}}|J} f^N(int(I)) = \mathcal{S}^1$.

One cannot expect memory loss out of an arbitrary composition of maps. We therefore consider a natural topology on the space of piecewise C^2 expanding maps. Let g be a piecewise C^2 expanding map. We say f is in the ε neighborhood $\mathcal{U}_\varepsilon$ if the following conditions are true.

1. the points of discontinuity of f and g are close.

2. f and g are C^2 close in an appropriate sense.

The set of density functions considered in this case is

$$\mathcal{D} = \{\varphi \in \mathrm{BV}(\mathcal{S}^1, \mathbb{R}) : \varphi \geq 0, \int_{\mathcal{S}^1} \varphi(x)\mathrm{d}x = 1\}$$

Theorem 5.3.6. *[68] Let $\mathcal{E}$ be the set of piecewise C^2 expanding maps with the enveloping property and let $g \in \mathcal{E}$. There exist $\Lambda < 1$ and $\varepsilon > 0$ such that for all $f_i \in \mathcal{U}_\varepsilon(g)$ and $\varphi, \psi \in \mathcal{D}$, there exists $C_{(\varphi,\psi)} > 0$ such that for all $n \in \mathbb{Z}^+$, we have*

$$\int_{\mathcal{S}^1} |\mathcal{P}_{F_n}(\varphi) - \mathcal{P}_{F_n}(\psi)|\mathrm{d}m \leq C_{(\varphi,\psi)}\Lambda^n.$$

5.4 Conditional memory loss

Open dynamical systems are considered as systems with holes in the phase space. The trajectories are considered until they fall into the holes. The study of statistical properties of open systems was introduced by Pianigiani and Yorke in [73], wherein they asked the following questions: Consider a particle on a billiard table whose dynamics are chaotic. Suppose a small hole is made in the table.

1. What are the statistical properties in this system ?

2. If one starts with an initial distribution μ_0 and μ_n represents the normalized distribution at time n, does μ_n convergence to some μ independent of μ_0?

Such a measure μ, if it is well-defined is called *conditionally invariant measure*. The existence and statistical properties of such measure for equilibrium open systems have been studied(see e.g. [52], [78]). For nonequilibrium systems, conditionally invariant measures do not exist in general. We study memory loss in nonequilibrium open context, a study motivated by following example.

Example 5.4.1. Open billiard with moving scatterers

Billiard dynamics are usually modelled by the motion of small particles inside a two-dimensional torus. The dynamical system is defined by the trajectories made by small particles in the domain $X = \mathbb{T}^2 - \bigcup_i \Gamma_i$,where the Γ_i are the convex subsets of $\mathbb{T}^2$ which represent the boundaries of scatterers inside the billiard table. The motion of the particle follows the rule that the angle of incidence is same as angle of the reflection at collisions. The scatterers Γ_i are often thought of as fixed. However it is more realistic to model them as slowly moving objects. This create nonequilibrium billiard. If one additionally introduces holes (which may vary with time), one has a nonequilibrium open billiard system.

In order to address the types of questions posed by Pianigiani and Yorke, we formally introduce nonequilibrium open systems and an appropriate notion of memory loss.

5.4.1 Memory loss for open nonequilibrium systems

We now introduce nonequilibrium open dynamical systems. Let X be a Riemannian manifold and let λ denote Riemannian volume (Lebesgue measure) on X. Consider a sequence of maps $(\hat{f}_i : X \to X)_{i=1}^{\infty}$. For $m \in \mathbb{N}$, define $\widehat{F}_m = \hat{f}_m \circ \cdots \circ \hat{f}_1$. We call the sequence $(\widehat{F}_m)_{m=1}^{\infty}$ a *nonequilibrium closed dynamical system.* Unlike the random dynamical systems setting, we do not assume that the $\hat{f}_i$ are drawn from a known distribution. Our setting is meant to model scenarios such as dynamical processes with time-varying parameters or dynamics in time-varying environments. An open system is produced by introducing holes. For $j \in \mathbb{N}$, let $H_j \subset X$. We call H_j the *hole at time* j. Informally, we create an open system from $(\widehat{F}_m)_{m=1}^{\infty}$ by tracking trajectories until they fall into a hole. Once a trajectory falls into a hole, it is deemed to have escaped. Formally, for $m \in \mathbb{N}$ define the *time-*m *survivor set* S_m by

$$S_m = X \setminus \bigcup_{i=1}^{m} (\widehat{F}_i)^{-1}(H_i).$$

Let F_m denote the restriction $\widehat{F}_m|S_m$; that is, F_m is defined on points with trajectories that have not fallen into a hole after m iterates. We call the pair $((F_m), (H_j))$ a *nonequilibrium open dynamical system.* We define a notion of memory loss for nonequilibrium open systems that is both statistical and conditional in nature as follows:

Definition 5.4.2. Let φ_0 and ψ_0 be two initial probability densities defined on X. Let φ_t and ψ_t denote the evolved densities under the action of the nonequilibrium open system. Since mass is allowed to escape through the holes, φ_t and ψ_t will not be probability densities in general: we expect $\|\varphi_t\|_{L^1(\lambda)} < 1$ and $\|\psi_t\|_{L^1(\lambda)} < 1$. We say that a nonequilibrium open system exhibits *conditional memory loss in the statistical sense* if for all initial densities φ_0 and ψ_0 chosen from a suitable class, we have

$$\lim_{t\to\infty} \left\| \frac{\varphi_t}{\|\varphi_t\|_{L^1(\lambda)}} - \frac{\psi_t}{\|\psi_t\|_{L^1(\lambda)}} \right\|_{L^1(\lambda)} = 0.$$

Ideally one explicitly estimates the rate of convergence as well.

We are motivated by the study of the statistical properties of open billiards with slowly moving scatterers. In the following Chapter, we study a one-dimensional setting as a nontrivial first step.

CHAPTER 6

Memory Loss for Nonequilibrium Open Dynamical Systems

In this chapter we establish conditional memory loss in the statistical sense for a class of nonequilibrium open systems generated by one-dimensional piecewise-differentiable expanding Lasota-Yorke maps. We work in this setting because it is simple enough to allow for a clear development of ideas yet complicated enough in that it has some of the features of more realistic settings. Using convex cones and a projective metric known as the Hilbert metric, we show that memory loss occurs at an exponential rate and we explicitly estimate this rate.

6.1 Setting and statement of results

6.1.1 Underlying closed dynamical systems

Let $[0,1]$ be the phase space on which our dynamical processes act. Let λ denote Lebesgue measure on $[0,1]$.

Definition 6.1.1. For $s < 1$, let $\mathcal{M}(s, K_2)$ denote the set of maps $\hat{g} : [0,1] \to [0,1]$ that satisfy the following hypotheses:

(a) there exists a finite partition $\mathcal{A}(\hat{g})$ of $[0,1]$ into subintervals such that for each $J \in \mathcal{A}(\hat{g})$, $\hat{g}$ is $\mathcal{C}^2$ on J and extends to a $\mathcal{C}^2$ function on $\overline{J}$;

(b) $\max_{J \in \mathcal{A}(\hat{g})} \sup_{x \in J} |\hat{g}'(x)|^{-1} \leqslant s$;

(c) $\max_{J \in \mathcal{A}(\hat{g})} \sup_{x \in J} |\hat{g}''(x)| \leqslant K_2$.

We now define δ-perturbations within $\mathcal{M}(s, K_2)$. Let $\hat{g} \in \mathcal{M}(s, K_2)$. Let $\Omega(\hat{g}) = \{0 = x_1, \ldots, x_k = 1\}$ be the set of partition points associated with $\mathcal{A}(\hat{g})$ and define $d_\Omega(\hat{g}) = \min_{1 \leqslant i \leqslant k-1} x_{i+1} - x_i$.

Definition 6.1.2. We say that $\hat{f} \in \mathcal{M}(s, K_2)$ is a δ-perturbation of $\hat{g} \in \mathcal{M}(s, K_2)$ if

(a) $\delta < \frac{1}{4} d_\Omega(\hat{g})$;

(b) $\Omega(\hat{f}) = \{0 = y_1, \ldots, y_k = 1\}$ where $|y_i - x_i| < \delta$ for every $1 \leqslant i \leqslant k$;

(c) if $\xi_{\hat{f}\hat{g}}$ maps each interval $[x_i, x_{i+1}]$ affinely onto $[y_i, y_{i+1}]$, then on every $J \in \mathcal{A}(\hat{g})$, we have

$$\left\| \hat{f} \circ \xi_{\hat{f}\hat{g}} - \hat{g} \right\|_{\mathcal{C}^2(J)} < \delta.$$

Let $\mathcal{N}(\hat{g}, \delta; s, K_2)$ denote the set of δ-perturbations of $\hat{g}$.

Remark 6.1.1. The set

$$\left\{ \mathcal{N}(\hat{g}, \delta; s, K_2) : \hat{g} \in \mathcal{M}(s, K_2), \quad \delta < \frac{1}{4} d_\Omega(\hat{g}) \right\}$$

is a basis for a topology on $\mathcal{M}(s, K_2)$.

Iterates of a single map $\hat{g} \in \mathcal{M}(s, K_2)$ do not necessarily exhibit memory loss. Indeed, memory loss is equivalent to measure-theoretic mixing in this context, and a single map $\hat{g} \in \mathcal{M}(s, K_2)$ may not even be ergodic. For this reason, we formulate an appropriate mixing condition.

Definition 6.1.3 (class $\mathcal{E}$)**.** Let $\zeta_1 \in (0, 1)$ and $\zeta_2 \in (1, \infty)$. We say that $\hat{g} : [0, 1] \to [0, 1]$ belongs to $\mathcal{E}(\zeta_1, \zeta_2)$ if the following hold.

(a) For every partition $\mathcal{Q}$ of $[0, 1]$ into subintervals of equal length, there exists a time $E(\mathcal{Q}, \zeta_1, \zeta_2)$ such that for every $J_1, J_2 \in \mathcal{Q}$, we have

$$\zeta_1 < \frac{\lambda(J_1 \cap \hat{g}^{-i}(J_2))}{\lambda(J_1)\lambda(J_2)} < \zeta_2 \tag{6.1}$$

for every $i \geqslant E(\mathcal{Q}, \zeta_1, \zeta_2)$.

(b) For every $x_j \in \Omega(\hat{g})$ and every $i \in \mathbb{N}$, we have

$$\operatorname{dist}\left(\lim_{z \to x_j^-} \hat{g}^i(z), \Omega(\hat{g}) \setminus \{0, 1\} \right) > 0, \qquad \operatorname{dist}\left(\lim_{z \to x_j^+} \hat{g}^i(z), \Omega(\hat{g}) \setminus \{0, 1\} \right) > 0.$$

For $x_j = 0$ ($x_j = 1$), only the limit from the right (left) is considered.

6.1.2 Nonequilibrium open dynamical systems

Start with a 'base map' $\hat{g} \in \mathcal{M}(s, K_2)$. Let $\delta > 0$ be small and consider a sequence of maps $(\hat{f}_i)_{i=1}^{\infty}$ in $\mathcal{N}(\hat{g}, \delta; s, K_2)$. For $m \in \mathbb{N}$, let $\widehat{F}_m = \hat{f}_m \circ \cdots \circ \hat{f}_1$. We call the sequence $(\widehat{F}_m)_{m=1}^{\infty}$ a *nonequilibrium closed dynamical system.*

We now introduce holes. For $j \in \mathbb{N}$, let $H_j \subset [0,1]$ denote the *hole at time* j. We assume that H_j consists of at most L pairwise-disjoint open subintervals $H_{j,k}$ of $[0,1]$. For $m \in \mathbb{N}$, define

$$S_m = [0,1] \setminus \bigcup_{i=1}^{m} (\widehat{F}_i)^{-1}(H_i).$$

We call S_m the *time-m survivor set.* Let F_m denote the restriction $\widehat{F}_m | S_m$. We call the pair $((F_m), (H_j))$ a *nonequilibrium open dynamical system.*

6.1.2.1 Densities and transfer operators

Let $\mathrm{BV}([0,1], \mathbb{R})$ denote the space of real-valued functions of bounded variation on $[0,1]$. The evolution of probability densities in $\mathrm{BV}([0,1], \mathbb{R})$ under the action of a nonequilibrium open dynamical system $((F_m), (H_j))$ is described by the family $(\mathcal{L}_{F_m})$ of transfer operators defined by

$$\mathcal{L}_{F_m}(\varphi)(x) = \sum_{z : F_m(z) = x} \frac{\varphi(z)}{|F_m'(z)|}$$

($\mathcal{L}_{F_m}(\varphi)(x) = 0$ if $F_m^{-1}(x) = \emptyset$). Of course, we expect to see $\|\mathcal{L}_{F_m}(\varphi)\|_{L^1(\lambda)} < \|\varphi\|_{L^1(\lambda)}$ in general, since mass will escape through the holes. We define operators $\mathcal{R}_{F_m}$ by renormalizing:

$$\mathcal{R}_{F_m}(\varphi) = \frac{\mathcal{L}_{F_m}(\varphi)}{\|\mathcal{L}_{F_m}(\varphi)\|_{L^1(\lambda)}}.$$

Notice that $\mathcal{R}_{F_m}$ is not linear. We are interested in the action of the sequence $(\mathcal{R}_{F_m})$ on the space

$$\mathcal{D} = \left\{\varphi \in \mathrm{BV}([0,1],\mathbb{R}) : \varphi \geqslant 0,\ \ \|\varphi\|_{L^1(\lambda)} = 1\right\}.$$

6.1.2.2 Main theorem

Theorem 6.1.4. *Let $\hat{g} \in \mathcal{M}(s, K_2) \cap \mathcal{E}(\zeta_1, \zeta_2)$ and let $L \in \mathbb{N}$. There exist $\delta_0 > 0$, $\varepsilon_0 > 0$, and $\Lambda < 1$ such that the following holds. Let $(\hat{f}_i)_{i=1}^{\infty}$ be any sequence of maps in $\mathcal{N}(\hat{g}, \delta_0; s, K_2)$ and let $(H_j)_{j=1}^{\infty}$ be any sequence of holes such that H_j consists of at most L pairwise-disjoint open intervals and $\lambda(H_j) \leqslant \varepsilon_0$ for every $j \in \mathbb{N}$. The resultant nonequilibrium open dynamical system $((F_m), (H_j))$ exhibits conditional memory loss in the following sense. There exists a convex cone $\mathcal{C}_a \subset \mathrm{BV}([0,1],\mathbb{R})$ and a constant $C_1 > 0$ such that for every $\varphi, \psi \in \mathcal{D} \cap \mathcal{C}_a$, we have*

$$\|\mathcal{R}_{F_m}(\varphi) - \mathcal{R}_{F_m}(\psi)\|_{L^1(\lambda)} \leqslant C_1 \Lambda^m \tag{6.2}$$

for all $m \in \mathbb{N}$.

Remark 6.1.2. See Section 6.2.2 and (6.15) for the definition of $\mathcal{C}_a$.

6.2 Proof of Theorem

6.2.1 A Lasota-Yorke inequality

We introduce several useful partitions of $[0,1]$. Let $\mathcal{Z}_1^{(n)} = \mathcal{Z}_1^{(n)}(\hat{f}_1, \ldots, \hat{f}_n)$ denote the dynamical partition for $\widehat{F}_n$. Let $\mathcal{Z}_2^{(n)}$ be the coarsest refinement of $\mathcal{Z}_1^{(n)}$ such that every element of $\mathcal{Z}_1^{(n)}$ is divided into subintervals of equal length and we have

$\lambda(J) \leqslant ns^{-1}K_2^{-1}$ for every $J \in \mathcal{Z}_2^{(n)}$. For $J \in \mathcal{Z}_2^{(n)}$, we have

$$\operatorname{Var}(|\widehat{F}_n'|^{-1}, J) = \int_J \left| \frac{\widehat{F}_n''(x)}{(\widehat{F}_n'(x))^2} \right| \mathrm{d}x \leqslant K_2 s^2 \lambda(J) \leqslant s^n. \tag{6.3}$$

Let $\mathcal{Z}_3^{(n)}$ be the coarsest refinement of $\mathcal{Z}_2^{(n)}$ such that for every $J \in \mathcal{Z}_3^{(n)}$, we have $J \subset S_n$ or $J \cap S_n = \emptyset$.

Proposition 6.2.1. *Let $\theta \in (s, 1)$ and $(\hat{f}_i)_{i=1}^{\infty}$ be any sequence of maps in $\mathcal{M}(s, K_2)$ and let $(H_j)_{j=1}^{\infty}$ be any sequence of holes such that H_j consists of at most L pairwise-disjoint open intervals. let $N_1 \in \mathbb{N}$ be such that*

$$\theta^{N_1} > 6s^{N_1}(LN_1 + 1). \tag{6.4}$$

For every $k \in \mathbb{N}$ and every nonnegative $\varphi \in \mathrm{BV}([0,1], \mathbb{R})$, we have

$$\begin{aligned} &\operatorname{Var}\left(\mathcal{L}_{F_{kN_1}}(\varphi), [0,1]\right) \\ &\qquad \leqslant \theta^{kN_1} \operatorname{Var}(\varphi, [0,1]) \\ &\qquad + \left((1-\theta^{N_1})^{-1} \cdot 5s^{N_1}(LN_1+1) \sup_{\widetilde{Z} \in \mathcal{Z}_2^{(N_1)}} \lambda(\widetilde{Z})^{-1} \right) \|\varphi\|_{L^1(\lambda)} . \end{aligned} \tag{6.5}$$

Proof of Proposition 6.2.1. Computing $\mathcal{L}_{F_n}(\varphi)$, we have

$$\mathcal{L}_{F_n}(\varphi) = \sum_{\substack{Z \in \mathcal{Z}_3^{(n)} \\ Z \subset S_n}} \mathcal{L}_{F_n}(\varphi \mathbf{1}_Z) \tag{6.6}$$

$$= \sum_{\substack{Z \in \mathcal{Z}_3^{(n)} \\ Z \subset S_n}} (\varphi \mathbf{1}_Z \cdot |F_n'|^{-1}) \circ (F_n|Z)^{-1}. \tag{6.7}$$

Therefore

$$\operatorname{Var}(\mathcal{L}_{F_n}(\varphi), [0,1]) \leqslant \sum_{\substack{Z \in \mathcal{Z}_3^{(n)} \\ Z \subset S_n}} \operatorname{Var}\left((\varphi \mathbf{1}_Z \cdot |F_n'|^{-1}) \circ (F_n|Z)^{-1}, [0,1]\right). \tag{6.8}$$

We estimate each term in the sum on the right side of (6.8). For $Z \subset \mathcal{Z}_3^{(n)}$ such that $Z \subset S_n$, let $\widetilde{Z} \in \mathcal{Z}_2^{(n)}$ be such that $Z \subset \widetilde{Z}$. For any such Z, we have

$$\begin{aligned}
\operatorname{Var}\left((\varphi \mathbf{1}_Z \cdot |F_n'|^{-1}) \circ (F_n|Z)^{-1}, [0,1]\right) &\leqslant \operatorname{Var}\left(\varphi |F_n'|^{-1}, \widetilde{Z}\right) + 2 \sup_{\widetilde{Z}} \varphi |F_n'|^{-1} \\
&\leqslant 3 \operatorname{Var}\left(\varphi |F_n'|^{-1}, \widetilde{Z}\right) + 2 \inf_{\widetilde{Z}} \varphi |F_n'|^{-1} \\
&\leqslant 3 \left[s^n \operatorname{Var}(\varphi, \widetilde{Z}) + (\sup_{\widetilde{Z}} \varphi) \operatorname{Var}\left(|F_n'|^{-1}, \widetilde{Z}\right) \right] \\
&\quad + 2 \inf_{\widetilde{Z}} \varphi |F_n'|^{-1} \\
&\leqslant 3 \left[s^n \operatorname{Var}(\varphi, \widetilde{Z}) + s^n (\sup_{\widetilde{Z}} \varphi) \right] + 2 s^n \inf_{\widetilde{Z}} \varphi \\
&\leqslant 6 s^n \operatorname{Var}(\varphi, \widetilde{Z}) + 5 s^n \inf_{\widetilde{Z}} \varphi.
\end{aligned} \tag{6.9}$$

Next observe that for every $\widetilde{Z} \in \mathcal{Z}_2^{(n)}$ we have

$$\# \left\{ Z \in \mathcal{Z}_3^{(n)} : Z \subset S_n \text{ and } Z \subset \widetilde{Z} \right\} \leqslant Ln + 1. \tag{6.10}$$

Estimates (6.8), (6.9), and (6.10) imply

$$\operatorname{Var}(\mathcal{L}_{F_n}(\varphi), [0,1]) \leqslant (Ln+1) \left(6 s^n \operatorname{Var}(\varphi, [0,1]) + 5 s^n (\sup_{\widetilde{Z} \in \mathcal{Z}_2^{(n)}} \lambda(\widetilde{Z})^{-1}) \, \|\varphi\|_{L^1(\lambda)} \right). \tag{6.11}$$

We choose $N_1 \in \mathbb{N}$ such that

$$\theta^{N_1} > 6 s^{N_1} (L N_1 + 1)$$

(see (6.4)), yielding

$$\operatorname{Var}(\mathcal{L}_{F_{N_1}}(\varphi), [0,1]) \leqslant \theta^{N_1} \operatorname{Var}(\varphi, [0,1]) + 5 s^{N_1} (L N_1 + 1) (\sup_{\widetilde{Z} \in \mathcal{Z}_2^{(N_1)}} \lambda(\widetilde{Z})^{-1}) \, \|\varphi\|_{L^1(\lambda)} . \tag{6.12}$$

We obtain the Lasota-Yorke estimate (6.5) by iterating (6.12). ■

6.2.2 Parameter selection

We prove Theorem 6.1.4 by studying the action of $\{\mathcal{L}_{F_m}\}$ on a suitable convex cone $\mathcal{C}_a$ of functions inside $\mathrm{BV}([0,1],\mathbb{R})$. We choose $\mathcal{Q}$ (recall Definition 6.1.3(a)) and introduce σ, T, and a such that (P1)–(P3) are simultaneously satisfied.

(P1) $0 < \sigma < 1$

(P2) $T \in \mathbb{N}$: choose such that T is a positive integer multiple of N_1, $T \geqslant E(\mathcal{Q},\zeta_1,\zeta_2)$, and $\theta^T < 1$. In view of (6.5), define

$$C_{\mathrm{LY}} = (1-\theta^{N_1})^{-1} \cdot 5s^{N_1}(LN_1+1) \sup_{\hat{f}_1,\ldots,\hat{f}_{N_1} \in \mathcal{N}(\hat{g},\delta_0;s,K_2)} \sup_{\widetilde{Z} \in \mathcal{Z}_2^{(N_1)}} \lambda(\widetilde{Z})^{-1}.$$

Definition 6.1.3(b) implies that $C_{\mathrm{LY}} < \infty$ if δ_0 is sufficiently small.

(P3) $a > 0$: the aperture of the cone $\mathcal{C}_a$. We choose a such that

$$\zeta_1 - \zeta_2 a \cdot \mathrm{diam}(\mathcal{Q}) > 0, \tag{6.13}$$

$$\frac{a\theta^T + C_{\mathrm{LY}}}{\zeta_1 - \zeta_2 a \cdot \mathrm{diam}(\mathcal{Q})} \leqslant \sigma a. \tag{6.14}$$

To see that (P1)–(P3) may be satisfied simultaneously, proceed in the following order:

(a) Choose T sufficiently large so that $\theta^T/(\zeta_1/2) < \sigma$.

(b) Choose a sufficiently large so that

$$\frac{a\theta^T + C_{\mathrm{LY}}}{\zeta_1/2} \leqslant \sigma a.$$

(c) Choose $\mathrm{diam}(\mathcal{Q})$ sufficiently small so that $\zeta_2 a \cdot \mathrm{diam}(\mathcal{Q}) \leqslant \zeta_1/2$.

(d) Increase T (if necessary) so that $T \geqslant E(\mathcal{Q},\zeta_1,\zeta_2)$.

6.2.3 Invariance of a suitable convex cone

Define

$$\mathcal{C}_a = \left\{\varphi \in L^1(\lambda) : \varphi \geqslant 0,\ \ \varphi \not\equiv 0,\ \ \mathrm{Var}(\varphi) \leqslant a\mathbb{E}[\varphi|\mathfrak{Q}]\right\}. \tag{6.15}$$

We study the action of $\mathcal{L}_{F_m}$ on $\mathcal{C}_a$. For positive integers $m > i$, define

$$\widehat{F}_{m,i} = \hat{f}_m \circ \hat{f}_{m-1} \circ \cdots \circ \hat{f}_i, \quad F_{m,i} = f_m \circ f_{m-1} \circ \cdots \circ f_i,$$

where f_k is the open system corresponding to $\hat{f}_k$ ($i \leqslant k \leqslant m$).

Lemma 6.2.2. *There exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for every $\varphi \in \mathcal{C}_a$ and $i \in \mathbb{N}$ we have*

$$(\zeta_1 - \zeta_2 a \cdot \mathrm{diam}(\mathfrak{Q})) \int_{[0,1]} \varphi \,\mathrm{d}\lambda \leqslant \mathbb{E}[\mathcal{L}_{F_{i+T-1,i}}(\varphi)|\mathfrak{Q}] \leqslant \zeta_2(1 + a \cdot \mathrm{diam}(\mathfrak{Q})) \int_{[0,1]} \varphi \,\mathrm{d}\lambda. \tag{6.16}$$

Proof of Lemma 6.2.2. First, choose δ_0 sufficiently small so that (6.1) holds for $\widehat{F}_{i+T-1,i}$ for all $i \in \mathbb{N}$. Second, choose ε_0 sufficiently small so that (6.1) holds for $F_{i+T-1,i}$ for all $i \in \mathbb{N}$.

Write $F = F_{i+T-1,i}$. For $x \in [0,1]$, let $Q(x)$ denote the element of $\mathfrak{Q}$ that contains x. We have

$$\begin{aligned}
\mathbb{E}[\mathcal{L}_F(\varphi)|\mathfrak{Q}](x) &= \frac{1}{\lambda(Q(x))} \int_{Q(x)} \mathcal{L}_F(\varphi) \,\mathrm{d}\lambda \\
&= \frac{1}{\lambda(Q(x))} \int_{F^{-1}(Q(x))} \varphi \,\mathrm{d}\lambda \\
&= \frac{1}{\lambda(Q(x))} \sum_{Q' \in \mathfrak{Q}} \int_{Q' \cap F^{-1}(Q(x))} \varphi(z) \,\mathrm{d}\lambda(z).
\end{aligned} \tag{6.17}$$

Bounding φ from below, for every $z \in Q' \cap F^{-1}(Q(x))$ we have

$$\begin{aligned}\varphi(z) &\geqslant \inf_{y\in Q'} \varphi(y) \\ &\geqslant \sup_{y\in Q'} \varphi(y) - \operatorname{Var}(\varphi, Q') \\ &\geqslant \frac{1}{\lambda(Q')} \int_{Q'} \varphi \,\mathrm{d}\lambda - \operatorname{Var}(\varphi, Q') \\ &= \frac{1}{\lambda(Q')} \left(\int_{Q'} \varphi \,\mathrm{d}\lambda - \lambda(Q') \operatorname{Var}(\varphi, Q') \right).\end{aligned} \tag{6.18}$$

Using (6.17), (6.18), and (6.1), we have

$$\begin{aligned}\mathbb{E}[\mathcal{L}_F(\varphi)|\mathcal{Q}](x) &\geqslant \frac{1}{\lambda(Q(x))} \sum_{Q'\in\mathcal{Q}} \int_{Q'\cap F^{-1}(Q(x))} \frac{1}{\lambda(Q')} \left(\int_{Q'} \varphi \,\mathrm{d}\lambda - \lambda(Q') \operatorname{Var}(\varphi, Q') \right) \mathrm{d}\lambda(z) \\ &= \sum_{Q'\in\mathcal{Q}} \frac{\lambda(Q' \cap F^{-1}(Q(x)))}{\lambda(Q(x))\lambda(Q')} \left(\int_{Q'} \varphi \,\mathrm{d}\lambda - \lambda(Q') \operatorname{Var}(\varphi, Q') \right) \\ &\geqslant \zeta_1 \int_{[0,1]} \varphi \,\mathrm{d}\lambda - \zeta_2 \cdot \operatorname{diam}(\mathcal{Q}) \cdot \operatorname{Var}(\varphi, [0,1]) \\ &\geqslant \zeta_1 \int_{[0,1]} \varphi \,\mathrm{d}\lambda - \zeta_2 a \cdot \operatorname{diam}(\mathcal{Q}) \cdot \int_{[0,1]} \varphi \,\mathrm{d}\lambda \\ &= (\zeta_1 - \zeta_2 a \cdot \operatorname{diam}(\mathcal{Q})) \int_{[0,1]} \varphi \,\mathrm{d}\lambda.\end{aligned} \tag{6.19}$$

The upper bound

$$\mathbb{E}[\mathcal{L}_F(\varphi)|\mathcal{Q}](x) \leqslant \zeta_2(1 + a \cdot \operatorname{diam}(\mathcal{Q})) \int_{[0,1]} \varphi \,\mathrm{d}\lambda$$

follows from an analogous line of reasoning. ■

Proposition 6.2.3. *In the setting of Lemma 6.2.2, for every $i \in \mathbb{N}$ we have*

$$\mathcal{L}_{F_{i+T-1,i}}(\mathcal{C}_a) \subset \mathcal{C}_{\sigma a}.$$

Proof of Proposition 6.2.3. Write $F = F_{i+T-1,i}$ and let $\varphi \in \mathcal{C}_a$. Using (6.5) and (6.16), we have

$$\begin{aligned} \operatorname{Var}(\mathcal{L}_F(\varphi), [0,1]) &\leqslant \theta^T \operatorname{Var}(\varphi, [0,1]) + C_{\mathrm{LY}} \|\varphi\|_{L^1(\lambda)} && (6.20) \\ &\leqslant (a\theta^T + C_{\mathrm{LY}}) \|\varphi\|_{L^1(\lambda)} && (6.21) \\ &\leqslant \frac{a\theta^T + C_{\mathrm{LY}}}{\zeta_1 - \zeta_2 a \cdot \operatorname{diam}(\mathcal{Q})} \mathbb{E}[\mathcal{L}_F(\varphi)|\mathcal{Q}] && (6.22) \\ &\leqslant \sigma a \mathbb{E}[\mathcal{L}_F(\varphi)|\mathcal{Q}]. && (6.23) \end{aligned}$$

■

6.2.4 Applying Hilbert metric method

Let $d_{\mathcal{C}_a}$ be the Hilbert metric defined in section 5.2.1 from chapter 4.

Proposition 6.2.4. *Assume the setting of Proposition 6.2.3. For every $i \in \mathbb{N}$ and for all $\varphi, \psi \in \mathcal{C}_a$, we have*

$$d_{\mathcal{C}_a}(\mathcal{L}_{F_{i+T-1,i}}(\varphi), \mathcal{L}_{F_{i+T-1,i}}(\psi)) \leqslant \Delta_0 := 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + 2\log\left(\frac{\zeta_2(1 + a \cdot \operatorname{diam}(\mathcal{Q}))}{\zeta_1 - \zeta_2 a \cdot \operatorname{diam}(\mathcal{Q})}\right). \tag{6.24}$$

Proof of Proposition 6.2.4. Let $\varphi^*, \psi^* \in \mathcal{C}_{\sigma a}$. Suppose $c > 0$. We have

$$\begin{aligned} \operatorname{Var}(c\varphi^* - \psi^*, [0,1]) &\leqslant c \operatorname{Var}(\varphi^*, [0,1]) + \operatorname{Var}(\psi^*, [0,1]) && (6.25) \\ &\leqslant c\sigma a \mathbb{E}[\varphi^*|\mathcal{Q}] + \sigma a \mathbb{E}[\psi^*|\mathcal{Q}]. && (6.26) \end{aligned}$$

Therefore $c\varphi^* - \psi^* \in \mathcal{C}_a$ if

$$c\sigma a \mathbb{E}[\varphi^*|\mathcal{Q}] + \sigma a \mathbb{E}[\psi^*|\mathcal{Q}] \leqslant a \mathbb{E}[c\varphi^* - \psi^*|\mathcal{Q}].$$

This is equivalent to

$$\left(\frac{1+\sigma}{1-\sigma}\right)\left(\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right) \leqslant c. \tag{6.27}$$

Arguing analogously, for $r > 0$ we have $\psi^* - r\varphi^* \in \mathcal{C}_a$ if

$$r \leqslant \left(\frac{1-\sigma}{1+\sigma}\right)\left(\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right). \tag{6.28}$$

Bounds (6.27) and (6.28) imply

$$\begin{aligned} d_{\mathcal{C}_a}(\varphi^*, \psi^*) &\leqslant \log\left(\left(\frac{1+\sigma}{1-\sigma}\right)\sup_{x\in[0,1]}\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right) - \log\left(\left(\frac{1-\sigma}{1+\sigma}\right)\inf_{x\in[0,1]}\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right) \\ &\leqslant 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + \log\left(\sup_{x\in[0,1]}\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right) - \log\left(\inf_{x\in[0,1]}\frac{\mathbb{E}[\psi^*|\mathcal{Q}]}{\mathbb{E}[\varphi^*|\mathcal{Q}]}\right). \end{aligned} \tag{6.29}$$

Proposition 6.2.3 and estimates (6.16) and (6.29) imply (6.24) with

$$\Delta_0 = 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + 2\log\left(\frac{\zeta_2(1 + a\cdot \mathrm{diam}(\mathcal{Q}))}{\zeta_1 - \zeta_2 a \cdot \mathrm{diam}(\mathcal{Q})}\right).$$

■

Corollary 6.2.5 (corollary of Proposition 6.2.4)**.** *Assume the setting of Proposition 6.2.4. For every $i \in \mathbb{N}$ and for all $\varphi, \psi \in \mathcal{C}_a$, we have*

$$d_{\mathcal{C}_a}(\mathcal{L}_{F_{i+T-1,i}}(\varphi), \mathcal{L}_{F_{i+T-1,i}}(\psi)) \leqslant \tanh\left(\frac{\Delta_0}{4}\right) d_{\mathcal{C}_a}(\varphi, \psi). \tag{6.30}$$

Proof of Corollary 6.2.5. The result follows directly from Birkhoff Theorem 5.2.4 from chapter 4 and Proposition 6.2.4. ■

We are nearly in position to derive (6.2). One additional ingredient is needed: a Lipschitz estimate involving $\mathcal{R}$.

Lemma 6.2.6. *Assume the setting of Corollary 6.2.5. There exists $C_{Lip} > 0$ such that for all integers n satisfying $1 \leqslant n < T$, for every $i \in \mathbb{N}$, and for all $\varphi, \psi \in \mathcal{D} \cap \mathcal{C}_a$, we have*

$$\left\|\mathcal{R}_{F_{i+n-1,i}}(\varphi) - \mathcal{R}_{F_{i+n-1,i}}(\psi)\right\|_{L^1(\lambda)} \leqslant C_{Lip} \left\|\varphi - \psi\right\|_{L^1(\lambda)}. \tag{6.31}$$

Proof of Lemma 6.2.6. Write $F = F_{i+n-1,i}$ and $\|\cdot\| = \|\cdot\|_{L^1(\lambda)}$. Let $\varphi, \psi \in \mathcal{D} \cap \mathcal{C}_a$. We have

$$\begin{aligned}
\|\mathcal{R}_F(\varphi) - \mathcal{R}_F(\psi)\| &= \left\| \frac{\mathcal{L}_F(\varphi)}{\|\mathcal{L}_F(\varphi)\|} - \frac{\mathcal{L}_F(\psi)}{\|\mathcal{L}_F(\psi)\|} \right\| && (6.32)\\
&= \left\| \frac{\mathcal{L}_F(\varphi)}{\|\mathcal{L}_F(\varphi)\|} - \frac{\mathcal{L}_F(\varphi)}{\|\mathcal{L}_F(\psi)\|} + \frac{\mathcal{L}_F(\varphi)}{\|\mathcal{L}_F(\psi)\|} - \frac{\mathcal{L}_F(\psi)}{\|\mathcal{L}_F(\psi)\|} \right\| && (6.33)\\
&\leqslant \frac{\left|\|\mathcal{L}_F(\psi)\| - \|\mathcal{L}_F(\varphi)\|\right|}{\|\mathcal{L}_F(\varphi)\| \cdot \|\mathcal{L}_F(\psi)\|} \|\mathcal{L}_F(\varphi)\| && (6.34)\\
&+ \frac{1}{\|\mathcal{L}_F(\psi)\|} \|\mathcal{L}_F(\varphi) - \mathcal{L}_F(\psi)\| && (6.35)\\
&\leqslant 2(\zeta_1 - \zeta_2 a \cdot \operatorname{diam}(\mathcal{Q}))^{-1} \|\mathcal{L}_F(\varphi) - \mathcal{L}_F(\psi)\| && (6.36)\\
&\leqslant 2(\zeta_1 - \zeta_2 a \cdot \operatorname{diam}(\mathcal{Q}))^{-1} \|\varphi - \psi\| && (6.37)
\end{aligned}$$

using (6.16). Set

$$C_{Lip} = 2(\zeta_1 - \zeta_2 a \cdot \operatorname{diam}(\mathcal{Q}))^{-1}.$$

■

We now derive (6.2). Write $\|\cdot\|_1$ for the L^1 norm. Let $\varphi, \psi \in \mathcal{D} \cap \mathcal{C}_a$. Let $m \in \mathbb{Z}^+$ and write $m = kT + n$ where $k \in \mathbb{Z}^+$ and $0 \leqslant n < T$. If $k \geqslant 1$, by using (6.31), Proposition 5.2.5, (6.30), and (6.24) respectively, we have

$$
\begin{aligned}
\|\mathcal{R}_{F_m}(\varphi) - \mathcal{R}_{F_m}(\psi)\|_1 &\leqslant C_{Lip} \|\mathcal{R}_{F_{kT}}(\varphi) - \mathcal{R}_{F_{kT}}(\psi)\|_1 \\
&\leqslant C_{Lip} \left(\exp\left(d_{\mathcal{C}_a}(\mathcal{R}_{F_{kT}}(\varphi), \mathcal{R}_{F_{kT}}(\psi))\right) - 1\right) \\
&= C_{Lip} \left(\exp\left(d_{\mathcal{C}_a}(\mathcal{L}_{F_{kT}}(\varphi), \mathcal{L}_{F_{kT}}(\psi))\right) - 1\right) \quad \text{(projectivity)} \\
&\leqslant C_{Lip} \left(\exp\left(\left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{k-1} d_{\mathcal{C}_a}(\mathcal{L}_{F_T}(\varphi), \mathcal{L}_{F_T}(\psi))\right) - 1\right) \\
&\leqslant C_{Lip} \Delta_0 e^{\Delta_0} \left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{k-1} \\
&\leqslant C_{Lip} \Delta_0 e^{\Delta_0} \tanh^{-2}\left(\frac{\Delta_0}{4}\right) \left(\left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{1/T}\right)^m
\end{aligned}
$$

Consequently, for any $m \in \mathbb{Z}^+$ we have

$$
\|\mathcal{R}_{F_m}(\varphi) - \mathcal{R}_{F_m}(\psi)\|_1 \leqslant C_{Lip} \max\{\Delta_0, 1\} e^{\Delta_0} \tanh^{-2}\left(\frac{\Delta_0}{4}\right) \left(\left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{1/T}\right)^m .
$$

This establishes (6.2) with

$$
C_1 = C_{Lip} \max\{\Delta_0, 1\} e^{\Delta_0} \tanh^{-2}\left(\frac{\Delta_0}{4}\right) \tag{6.38}
$$

$$
\Lambda = \left(\tanh\left(\frac{\Delta_0}{4}\right)\right)^{1/T} . \tag{6.39}
$$

CHAPTER 7

Discussion

We conclude the dissertation with some observations and possible future directions. Following future directions can arise from the first part of the dissertation.

- Our results apply to heteroclinic cycles wherein each saddle has a one-dimensional unstable manifold. What happen if saddles has more than one unstable manifolds?
- Do our results extend to infinite-dimensional dynamical systems?
- Can we formulate results for heteroclinic networks?
- We assume the existence and form of the global map in our settings. There can be detail study of existence and explicit computation of global maps relating to forcing functions in general. Numerical algorithms may be useful in this

context.

Our work on conditional memory loss suggests the follwing future directions

- Formulation of the conditional memory loss idea in other different interesting settings such as nonuniformly hyperbolic systems, skew product and lattice systems.
- Extension of the result to higher dimensions where the geometry is complicated due to presence of holes in the domain.

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