Superconvergence of discontinuous Galerkin method for hyperbolic problems

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Supported by NSF and UH.
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Feb. 2012, University of Houston

Outline

- Part I. Introduction
  - Hyperbolic conservation laws
  - DG method: formulation, implementation, properties
- Part II. Superconvergence of DG
  - Review of literature: negative norm, post-processed solution, Radau projection
  - Fourier analysis for linear problem
    - eigenvalues
    - eigenvectors
  - Simulation result
  - Ongoing and future work

DG for hyperbolic conservation laws

Hyperbolic conservation laws

$$\begin{cases} \mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \\ \mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0(\mathbf{x}) \end{cases} \quad (1)$$

- $\mathbf{u}$: conserved quantities

$$\frac{d}{dt} \int \mathbf{u} dx = 0.$$

- $\mathbf{f}$: flux functions
- For example, Euler equations for fluid dynamics is a system of three equations in the form of (1) with

$$\mathbf{u} = (\rho, m, E)'$$

representing the conservation of mass, momentum and energy of the system.

Features of solutions for hyperbolic equations:

- solution is constant along characteristics

$$\frac{dx}{dt} = f'(u)$$

- when $f(u)$ is linear, e.g. $f(u) = u$
  – characteristics: $\frac{dx}{dt} = 1$
  – linear advection of initial data from left to right with speed 1.
- when $f(u)$ is nonlinear, e.g. $f(u) = u^2$
  – characteristics: $\frac{dx}{dt} = u(x(t = t_0), t = 0)$
  – depending on the sign of initial data, characteristics go to different directions
  – when characteristics run into each other: development of discontinuities even from smooth initial data

Approximation space for DG

Define the approximation space as

$$V_h^k = \left\{ v : v|_{I_j} \in P^k(I_j);\ 1 \le j \le N \right\} \tag{2}$$

based on a partition of the computational domain

$$[a, b] = \cup I_j = \cup [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}].$$

- k is the polynomial degree, h is the mesh size
- Functions in V_h^k is in general discontinuous across the cell boundaries.
- Note that solutions for hyperbolic problem might develop discontinuities/shocks anyway.

DG for hyperbolic equation

A semi-discrete DG [1] formulation for 1-D hyperbolic problem (1) is to find a piecewise polynomial function $u_h \in V_h^k$, s.t.

$$\frac{d}{dt}\int_{I_j} u_h v dx = \int_{I_j} f(u_h) v_x dx - \hat{f}_{j+1/2} v|_{x_{j+1/2}} + \hat{f}_{j-1/2} v|_{x_{j-1/2}}, \quad (3)$$

$$\forall v \in P^k(I_j).$$

[1] Cockburn and Shu, 80's

- The numerical flux function

$$\hat{f}_{j+1/2} = \hat{f}(u^-_{j+1/2}, u^+_{j+1/2}),$$

  is designed based on how information propagates along characteristics. Especially,

$$\hat{f}(\uparrow, \downarrow)$$

  For example, Godunov flux, Lax-Friedrichs flux, $\cdots$
- Strong stability preserving Runge-Kutta method is used to evolve the solution in time.

Implementation of DG

1. Choose a set of basis for $P^k(I_j)$ on I_j

$$\{\phi_1(\xi), \cdots, \phi_{k+1}(\xi)\}, \quad \xi = \frac{x - x_j}{h}$$

For example

- monomials $\{1, \xi, \cdots \xi^k\}$
- Legendre polynomials
- nodal basis

$$\{L_i(\xi) = \text{Lagrangian polynomial}, \quad i = 1, \cdots k+1\}$$

2. Let

$$u_h(x,t) = \sum_{i=1}^{k+1} \hat{u}_i(t)\phi_i(\xi)$$

$$v = \phi_l(\xi), \quad l = 1, \cdots k+1$$

3. Let $\mathbf{u}_j = (\hat{u}_1, \cdots \hat{u}_{k+1})'$

$$\frac{d}{dt}\mathbf{u}_j = \mathbf{f}(\mathbf{u}_{j-1}, \mathbf{u}_j, \mathbf{u}_{j+1})$$

e.g. for linear problem $(f(u) = u)$,

$$\frac{d}{dt}\mathbf{u}_j = \frac{1}{h}(A\mathbf{u}_j + B\mathbf{u}_{j-1})$$

Properties of DG

- compact and flexible in handling complicated geometry
- h-p adaptivity
- maximum principle preserving limiters
- L^2 stability for nonlinear problems
- L^2 error estimate for linear problems
- super convergence

L^2 stability of DG

L^2 stability [2]:

$$\|u_h(T)\|_2^2 \le \|u_h(0)\|_2^2.$$

Specifically,

$$\|u_h(T)\|_2^2 + \Theta_T(u_h) \le \|u_h(0)\|_2^2,$$

with

$$\Theta_T(u_h) = \alpha \int_0^T \sum_j [u_h(t)]_{j+\frac{1}{2}}^2 dt.$$

$\alpha = \max_u |f'(u)|$.

[2] Jiang and Shu, 90's

Error estimates of DG for linear equation

Let $e = u - u_h$

$$\|e(T)\|_2 \leq C\|u_0\|_{H^{k+2}} h^{k+1}$$

Super convergence of DG

Superconvergence of DG

For linear problem

- Negative norm and post-processed solution (Cockburn et. al. 2003)
- Radau projection and time evolution of error (Cheng and Shu, 2008)
- Radau and downwind points (Adjerid et. al. 2001)
- Dispersion and dissipation error of physically relevant eigenvalues in Fourier analysis (Ainsworth, 2004)

Negative norm and post-processed solution

- L^2 norm

$$\|e(T)\|_2 \leq C\|u_0\|_{H^{k+2}} h^{k+1}$$

- negative norm

$$\|e(T)\|_{-(k+1)} \leq \|u_0\|_{H^{k+1}} h^{2k+1}$$

with negative norm defined by

$$\|u\|_{-l} = sup_{v \in C_0^\infty} \frac{\int_\Omega uvdx}{\|v\|_{l,\Omega}}$$

- Post-processed solution via kernel convolution: $u_h^\star = K * u_h$

$$\|u(T) - u_h^\star(T)\|_0 \leq C h^{2k+1}$$

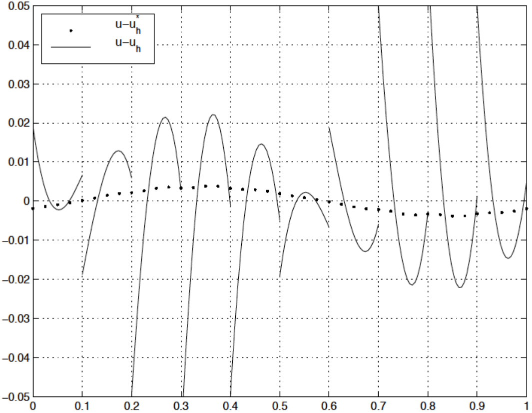

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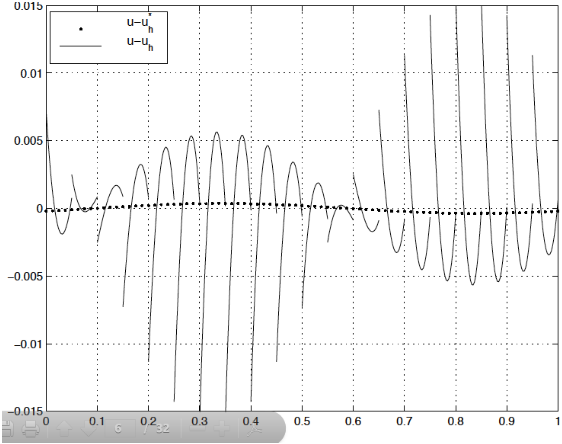
Figure: $e(T = 0.1)$ of DG P^2 solution for linear advection equation. $N = 10$.

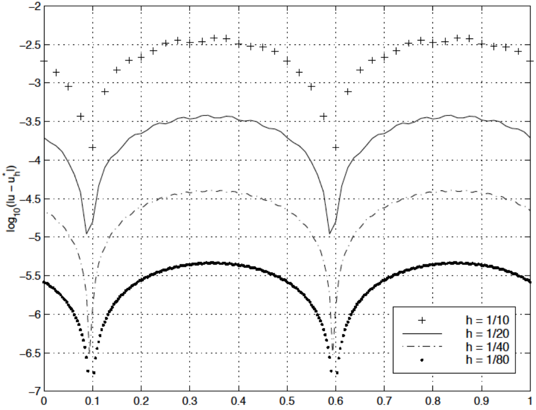
[3]Enhanced Accuracy by Post-Processing for Finite Element Methods for Hyperbolic Equations, by Bernardo Cockburn, Mitchell Luskin, Chi-Wang Shu and Endre Sli



[4]

Figure: $e(T = 0.1)$ of DG P^2 solution for linear advection equation. $N = 20$.

[4] Enhanced Accuracy by Post-Processing for Finite Element Methods for Hyperbolic Equations, by Bernardo Cockburn, Mitchell Luskin, Chi-Wang Shu and Endre Sli



[5]

Figure: $e(T = 0.1)$ of DG P^2 solution for linear advection equation. $\|u - u_h^\star\| = \mathcal{O}(h^{2k+1})$.

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[5] Enhanced Accuracy by Post-Processing for Finite Element Methods for Hyperbolic Equations, by Bernardo Cockburn, Mitchell Luskin, Chi-Wang Shu and Endre Sli

Radau projection

Let

- $P_h^- u$ be polynomials interpolating u at Radau points on each element
- $\bar{e} = P_h^- u - u_h$

Then

- $$\|\bar{e}(T)\|_2 \le C_1 h^{k+2} T \tag{4}$$
- $$\begin{aligned} \|e(T)\|_2 &\le \|u - P_h^- u\|_2 + \|\bar{e}(T)\|_2 \\ &\le C_2 h^{k+1} + C_1 h^{k+2} T \end{aligned} \tag{5}$$

Table 2.5

The errors $\bar{e}$ and e for Example 1a when using P^2 polynomials and SSP ninth-order time discretization on a uniform mesh of N cells ($CFL = 0.1$)

	N	$T=1$		$T=100$		$T=1000$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order
$\bar{e}$	20	4.17E−06	-	3.02E−05	-	2.99E−04	-
	40	2.62E−07	3.99	9.74E−07	4.95	9.38E−06	4.99
	80	1.64E−08	4.00	3.36E−08	4.86	2.94E−07	5.00
	160	1.02E−09	4.00	1.37E−09	4.61	9.91E−09	4.89
e	20	1.07E−04	-	1.11E−04	-	3.18E−04	-
	40	1.34E−05	3.00	1.34E−05	3.05	1.63E−05	4.28
	80	1.67E−06	3.00	1.67E−06	3.00	1.70E−06	3.28
	160	2.09E−07	3.00	2.09E−07	3.00	2.09E−07	3.02

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[6] Superconvergence and time evolution of discontinuous Galerkin finite element solutions, by Yingda Cheng and Chi-Wang Shu

- The numerical solution u_h is closer to $P_h^- u$ than to the exact solution itself.
- When $T = o(\frac{1}{h})$, $C_2 h^{k+1}$ is the dominant term:
  – time independent and of order $k+1$.
- When $T = \mathcal{O}(\frac{1}{h})$, $C_1 h^{k+2} T$ is the dominant term:
  – linearly grow with time and of order $k+2$.

From equation (4), it is expected that $\bar{e}$ is on the order of $k+2=4$. However, superior performance (5th order) is observed. Sharper estimate is yet to be explored?

Explore super convergence via Fourier analysis

Fourier/Von Neumann analysis

- is an approach to analyze stability and accuracy of numerical schemes
- is restrictive
  - to problems with periodic b.c.
  - to schemes with uniform mesh
- may serve as
  - a sufficient condition as instability of a numerical algorithm
  - a guide for error estimate for more general setting

Fourier analysis for linear equation

Consider linear equation

$$\begin{cases} u_t + u_x = 0, \quad x \in [0, 2\pi], t > 0 \\ u(x, 0) = u_0(x), \; x \in [0, 2\pi] \end{cases}$$

In Fourier space, assume

$$u(x, t) = \sum_{\omega} \hat{u}_{\omega}(t) \exp(i\omega x)$$

then

$$\frac{d}{dt}\hat{u}_{\omega}(t) + i\omega\hat{u}_{\omega}(t) = 0 \Rightarrow \hat{u}_{\omega}(t) = \exp(-i\omega t)\hat{u}_{\omega}(0)$$

WLOG, consider a single mode $exp(i\omega x)$.

Fourier analysis for DG

Based on the assumption of uniform mesh and initial data $u(x,0) = exp(i\omega x)$, we assume on each element I_j

$$\mathbf{u}_j = \mathbf{u}(t) exp(i\omega x_j). \quad (6)$$

- $\mathbf{u} = (\hat{u}_1, \cdots \hat{u}_{k+1})'$ is the degree of freedom on each cell
- the spatial dependence is from $exp(i\omega x_j)$. Especially, between neighboring cells, the difference is the ratio $exp(i\omega h)$.

Substituting (6) into the DG scheme, the coefficient vector satisfies the following ODE system

$$\mathbf{u}'(t) = G\mathbf{u}(t),$$

where G is the amplification matrix of size $(k+1) \times (k+1)$

$$G = \frac{1}{h}(A + Be^{-i\xi}), \quad \xi = \omega h.$$

Let

- eigenvalues of G as

$$\lambda_1, \cdots, \lambda_{k+1}$$

- the corresponding eigenvectors as

$$\tilde{V}_1, \cdots, \tilde{V}_{k+1}$$

Then

$$\begin{aligned}
\mathbf{u}(t) &= C_1\, e^{\lambda_1 t}\, \tilde{V}_1 + \cdots + C_{k+1}\, e^{\lambda_{k+1} t}\, \tilde{V}_{k+1}, \\
&= e^{\lambda_1 t}\, V_1 + \cdots + e^{\lambda_{k+1} t}\, V_{k+1}
\end{aligned}$$

where the coefficients $C_1, \cdots, C_{k+1}$ determined by the initial condition and $V_l = C_l \tilde{V}_l$.

Eigenvalues of G

(k+1) eigenvalues

- one of which is physically relevant, approximating $i\omega$ with high order accuracy [7]
  - order $2k+1$ dissipation error
  - order $2k+2$ dispersion error
- k of which has large negative real part ($O(-\frac{1}{h})$).
  – This indicates that the corresponding eigenvector will be damped out exponentially fast.

Remark

Eigenvalues are independent of choices of basis in DG implementation.

[7] Ainsworth, 04'

Symbolic analysis on eigenvalues

- P^1

$$\lambda_1 = -ik - \frac{k^4}{72}h^3 + O(h^4)$$

$$\lambda_2 = -\frac{6}{h} + 3ik + k^2 h + O(h^2)$$

- P^2

$$\lambda_1 = -ik - \frac{k^6}{7200}h^5 + O(h^6)$$

$$\lambda_2 = \frac{-3 + \sqrt{51}i}{h} + O(1)$$

$$\lambda_3 = \frac{-3 - \sqrt{51}i}{h} + O(1)$$

- P^3

$$\lambda_1 = -ik - 7.1 \times 10^{-7} k^8 h^7 + O(h^8)$$

$$\lambda_2 = \frac{-0.42 + 6.61i}{h} + O(1)$$

$$\lambda_3 = \frac{-0.42 - 6.61i}{h} + O(1)$$

$$\lambda_4 = -\frac{19.15}{h} + O(1)$$

Eigenvectors of G

With *Lagrangian basis functions based on Radau points* on each element,

$$V_l = \mathcal{O}(h^{k+2}), \quad l = 2, \cdots k+1$$

$$\|V_1 - \mathbf{u}(t=0)\|_\infty \le \sum_{l=2}^{k+1} \|V_l\|_\infty = \mathcal{O}(h^{k+2})$$

Symbolic analysis on eigenvectors

- P^1

$$V_2 = \begin{pmatrix} -\frac{ik^3}{162}h^3 + O(h^4) \\ \\ \frac{ik^3}{54}h^3 + O(h^4) \end{pmatrix} \Rightarrow \|V_2\|_\infty = \mathcal{O}(h^3)$$

- P^2: $V_{2,3} =$

$$
\begin{pmatrix}
-\dfrac{\left(153+408\sqrt{6}\pm i18\sqrt{34}\mp i29\sqrt{51}\right)k^4}{2040000}h^4+O(h^5) \\
-\dfrac{\left(153-408\sqrt{6}\mp i18\sqrt{34}+\mp i29\sqrt{51}\right)k^4}{2040000}h^4+O(h^5) \\
-\dfrac{ik^4}{160\sqrt{51}}h^4+O(h^5)
\end{pmatrix}
$$

- P^3

$$V_2 = \begin{pmatrix} (2.13 \times 10^{-5} + i1.19 \times 10^{-5})k^5h^5 + O(h^6) \\ (1.55 \times 10^{-6} - i1.86 \times 10^{-5})k^5h^5 + O(h^6) \\ (-1.73 \times 10^{-5} + i9.61 \times 10^{-6})k^5h^5 + O(h^6) \\ (6.53 \times 10^{-6} + i2.31 \times 10^{-5})k^5h^5 + O(h^6) \end{pmatrix}$$

$$V_3 = \begin{pmatrix} (-2.13 \times 10^{-5} + i1.19 \times 10^{-5})k^5h^5 + O(h^6) \\ (-1.55 \times 10^{-6} - i1.86 \times 10^{-5})k^5h^5 + O(h^6) \\ (1.73 \times 10^{-5} + i9.61 \times 10^{-6})k^5h^5 + O(h^6) \\ (-6.53 \times 10^{-6} + i2.31 \times 10^{-5})k^5h^5 + O(h^6) \end{pmatrix}$$

$$V_4 = \begin{pmatrix} 2.20 \times 10^{-5} ik^5h^5 + O(h^6) \\ -1.09 \times 10^{-5} ik^5h^5 + O(h^6) \\ 6.85 \times 10^{-6} ik^5h^5 + O(h^6) \\ -4.62 \times 10^{-5} ik^5h^5 + O(h^6) \end{pmatrix}$$

Error of DG solution

Proposition

Consider DG with P^k ($k \leq 3$) solution space for linear hyperbolic equation $u_t + u_x = 0$ with uniform mesh, periodic boundary condition. Let $\vec{u}$ and $\vec{u}_h$ be the point values of exact and numerical solution at right Radau points respectively. Let $\vec{e} = \vec{u} - \vec{u}_h$. Then

$$\|\vec{e}(T)\| = \mathcal{O}(h^{2k+1})T + \mathcal{O}(h^{k+2})$$

Proof. Consider Lagrangian basis functions at Radau points as basis functions on each DG element,

$$
\begin{aligned}
\|\vec{e}(T)\| &= \|\vec{u}(T) - \vec{u}_h(T)\| \\
&= \|(exp(i\omega T)\vec{u}(0) - \sum_{l=1}^{k+1} exp(\lambda_l t) V_l\| \\
&\leq \|(exp(i\omega T) - exp(\lambda_1 T)) V_1\| \\
&\quad + \sum_{l=2}^{k+1} \|(exp(i\omega t) - exp(\lambda_l t)) V_l\| \\
&\leq |exp(i\omega T) - exp(\lambda_1 T)| \|V_1\| \\
&\quad + \sum_{l=2}^{k+1} (1 + |exp(\lambda_l t)|) \|V_l\| \\
&= \mathcal{O}(h^{2k+1}) T \|V_1\| + \sum_{l=2}^{k+1} (1 + exp(-\frac{1}{h})) \|V_l\| \\
&= \mathcal{O}(h^{2k+1}) T + \mathcal{O}(h^{k+2}) \qquad \square
\end{aligned}
$$

Remark

The error of the DG solution can be decomposed as two parts:

1. the dispersion and dissipation error of the physically relevant eigenvalue; this part of error will grow linearly in time and is of order $2k + 1$
2. projection error, that is, there exists a special projection of the solution (V_1) such that the numerical solution is much closer to the special projection of exact solution, than the exact solution itself; the magnitude of this part of error will not grow in time.

Remark

1. When $T = o(\frac{1}{h^k})$, $\mathcal{O}(h^{k+2})$ is the dominant term: time independent and of order $k+2$.
2. When $T = \mathcal{O}(\frac{1}{h^k})$, $\mathcal{O}(h^{2k+1})T$ is the dominant term: linearly grow with time and of order $2k+1$.

- The special projection V_1 is of order $\mathcal{O}(h^{k+2})$ close to the Radau projection of the solution.
- However, the exact form of such special projection is not known.
- To obtain V_1, one can use DG to integrate the solution to 2π. After time integration, the eigenvectors corresponding to unphysical eigenvalues will be damped out exponentially.

Corollary

Consider DG with P^k ($k \leq 3$) solution space for linear hyperbolic equation $u_t + u_x = 0$ with uniform mesh, periodic boundary condition. Let n be a positive integer.

$$\|\vec{u}_h(2n\pi) - \vec{u}_h(2\pi)\| = \mathcal{O}(h^{2k+1})(n-1)$$

Simulation results: DG solution for $u_t + u_x = 0$

Table: Linear advection $u_t + u_x = 0$. The L^2 error and order of accuracy of $\bar{e}_1 = \|u_h(x, t = 4\pi) - u_h(x, t = 2\pi)\|_2$. Uniform mesh.

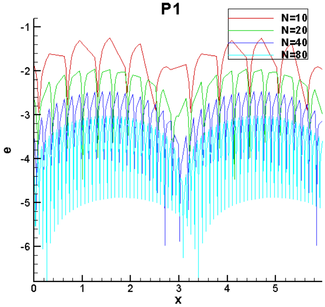
	P^1		P^2		P^3	
mesh	L^2 error	order	L^2 error	order	L^2 error	order
10	2.07E-02	–	8.35E-05	–	2.75E-06	–
20	2.66E-03	2.96	2.66E-06	4.97	4.50E-09	9.26
30	8.00E-04	2.97	3.51E-07	5.00	8.97E-11	9.65
40	3.37E-04	3.00	8.34E-08	4.99	1.04E-11	7.48
50	1.73E-04	2.99	2.73E-08	5.00	2.11E-12	7.16

Table: Linear advection $u_t + u_x = 0$. The L^2 error and order of accuracy of $\bar{e}_2 = \|u_h(x, t = 6\pi) - u_h(x, t = 2\pi)\|_2$. Uniform mesh.

	P^1		P^2		P^3	
mesh	L^2 error	order	L^2 error	order	L^2 error	order
10	4.10E-02	–	1.67E-04	–	2.89E-06	–
20	5.32E-03	2.95	5.32E-06	4.97	5.75E-09	8.97
30	1.60E-03	2.97	7.02E-07	5.00	1.68E-10	8.72
40	6.74E-04	3.00	1.67E-07	4.99	2.08E-11	7.25
50	3.46E-04	2.99	5.46E-08	5.00	4.25E-12	7.13

Table: Linear advection $u_t + u_x = 0$. The L^2 error and order of accuracy of $\bar{e}_1$. Nonuniform mesh with 10% random perturbation.

	P^1		P^2		P^3	
mesh	L^2 error	order	L^2 error	order	L^2 error	order
10	4.22E-02	–	2.98E-04	–	4.31E-06	–
20	2.70E-03	3.97	2.95E-06	6.66	3.79E-09	10.15
30	8.07E-04	2.98	3.88E-07	5.00	5.10E-10	4.95
40	3.40E-04	3.00	9.66E-08	4.83	1.62E-10	3.99
50	1.75E-04	2.99	2.97E-08	5.29	4.96E-11	5.30



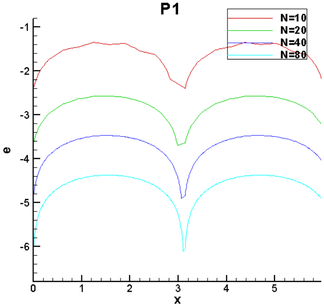


Figure: DG with P^1 for linear hyperbolic problem. Left: error of DG solution $|u - u_h|$ at $T = 4\pi$; right: error of $\|u_h(4\pi) - u_h(2\pi)\|$.

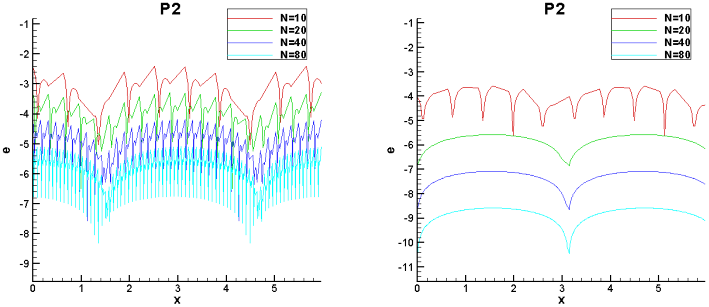


Figure: DG with P^2 for linear hyperbolic problem. Left: error of DG solution $|u - u_h|$ at $T = 4\pi$; right: error of $\|u_h(4\pi) - u_h(2\pi)\|$.

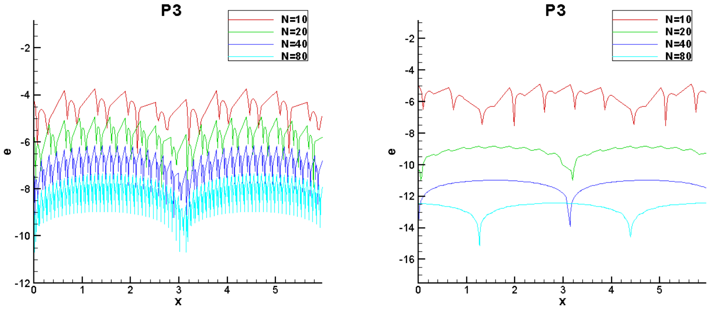


Figure: DG with P^3 for linear hyperbolic problem. Left: error of DG solution $|u - u_h|$ at $T = 4\pi$; right: error of $\|u_h(4\pi) - u_h(2\pi)\|$.

DG solution for linear variable coefficient equation

Consider

$$\begin{cases} u_t + (a(x)u)_x = b(x,t), & x \in [0, 2\pi] \\ u(x,0) = \sin(x) \end{cases}$$

with

$$a(x) = \sin(x) + 2,$$

$$b(x,t) = (\sin(x) + 3)\cos(x+t) + \cos(x)\sin(x+t),$$

and periodic boundary condition. The exact solution is

$$u(x,t) = \sin(x+t).$$

SSPRK(5,4) is used for the time integration.

Table: Linear variable coefficient problem. The L^2 error and order of accuracy of $\bar{e}_1 = \|u_h(x, t = 4\pi) - u_h(x, t = 2\pi)\|_2$. Uniform mesh.

	P^1		P^2		P^3	
mesh	L^2 error	order	L^2 error	order	L^2 error	order
20	5.00E-04	–	9.43E-07	–	8.70E-08	–
30	1.68E-04	2.68	1.24E-07	5.00	5.66E-09	6.74
40	7.37E-05	2.87	2.95E-08	5.00	7.42E-10	7.06
50	3.83E-05	2.93	9.67E-09	5.00	1.20E-10	8.17
60	2.23E-05	2.96	3.88E-09	5.00	2.26E-11	9.15

DG solution for nonlinear problem

Consider

$$\begin{cases} u_t + (u^3)_x = b(x,t), & x \in [0, 2\pi] \\ u(x,0) = \sin(x) \end{cases}$$

with periodic boundary condition.

$$b(x,t) = (1 + 3\sin^2(x+t))\cos(x+t)$$

The exact solution is $u(x,t) = \sin(x+t)$. SSPRK(5,4) is used for the time integration.

Table: Nonlinear Problem. The L^2 error and order of accuracy of $\bar{e}_1$. Uniform mesh.

	P^1		P^2		P^3	
mesh	L^2 error	order	L^2 error	order	L^2 error	order
20	7.91E-05	–	7.55E-06	–	7.90E-08	–
40	3.52E-06	4.49	2.17E-07	5.12	3.43E-09	4.53
60	4.92E-07	4.85	2.72E-08	5.12	6.22E-10	4.21
80	1.10E-07	5.21	4.57E-09	6.20	1.81E-10	4.29
100	3.25E-08	5.47	9.97E-10	6.82	6.65E-11	4.49

Future work

- Theoretical, rather than symbolic, proof of results.
- Seek for the explicit form of special projection V_1.
- Answers questions for non-uniform mesh and nonlinear problem.
- Extend this result to local DG.

THANK YOU!