A Real Analysis Prelim Question Bank

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1 Introduction

Students are not expected to have taken graduate topology, so in most questions involving topologies, the student can assume if they wish that the space is an open or closed set in $\mathbb{R}^n$.

This document is rather lengthy. The problems in this problem bank came from several different sources. Some problems may not be relevant for the Prelim you take, and different instructors may emphasize different topics. Thus it is far from necessary to know the answers to every question in this list in order to pass the preliminary exam. However, you should expect the exam to require a general mastery of the topics covered here. This problem bank provides a sampling of questions from the topics in the Real Analysis prelim syllabus at UH, which may be found on the department website.

This is not a comprehensive list of all possible questions. One can easily find online many more practice questions for real variable/analysis prelims from other schools. Any given instructor may include questions (both in the course and on the prelim) that concern the topics here but are not explicitly stated as problems here. For example, the emphasis below is often on 'homework-type exercises', whereas the prelim you take may have much more emphasis on definitions, statements of results, or proofs.

2 Some recommended texts

The Prelim syllabus also lists the following recommended textbooks, which the student could use together with their course materials, to find many additional practice problems.

(1) S. Axler, "Measure, Integration Real Analysis", 2020 (free from author's website)

(2) R. Bass, "Real Analysis for Graduate Students", 2014 (free from author's website)

(3) G. Folland, "Real Analysis: Modern Techniques and their Applications", 1999

(4) W. Rudin, "Real and Complex Analysis", 1987

(5) D. Salamon, "Measure and Integration", 2016

(6) J. Yeh, "Theory of Measure and Integration", 2006

3 Measures

3.1 Algebra, σ-algebra, measurable space, measurable set, Borel σ-algebra

1. What is a σ-algebra? What is a measure? What is a measure space? What is the Borel σ-algebra on a topological space? What is a Borel set? Show that the set $[0,1) \cup \mathbb{Q}$ is a Borel set.

2. Find an example of a set X and two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, each consisting of subsets of X, such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra.

3. Let $(Y, \mathcal{A})$ be a measurable space and let $f\colon X \to Y$ be a function, but do not assume that f is one-to-one. Define $f^{-1}(\mathcal{A}) := \{f^{-1}(E) : E \in \mathcal{A}\}$. Prove that $f^{-1}(\mathcal{A})$ is a σ-algebra on X.

4. Suppose that $(X, \mathcal{A})$ is a measurable space and that $f\colon X \to Y$ is a function. Show that $\mathcal{B} := \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y.

5. Show that if f is continuous then $f^{-1}(B)$ is Borel for Borel sets B.

3.2 Measures: σ-finite, finite, probability, complete, Borel.

1. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{A})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{A})$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, $F \in \mathcal{A}$, and define $\nu(E) := \mu(E \cap F)$ for $E \in \mathcal{A}$. Then ν is a measure.

3. If $(X, \mathcal{A}, \mu)$ is a measure space and $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{A}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$, provided that $\mu(\bigcup_{j=1}^{\infty} E_j)$ is finite.

4. If $(X, \mathcal{A}, \mu)$ is a measure space and $E, F \in \mathcal{A}$, then $\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cup F)$.

5. Suppose that μ is a finite measure on X, that $\alpha > 0$, and that $E_1, E_2, \ldots$ are measurable sets with $\mu(E_n) \geq \alpha$ for all $n \in \mathbf{N}$. Prove that there is a point in X which is in infinitely many of the E_n.

6. Let $(Y, \mathcal{A}, \mu)$ be a measure space and $f\colon X \to Y$ be a bijective function. Show that $\nu(E) := \mu(f(E))$ defines a measure ν on $(X, f^{-1}(\mathcal{A}))$.

7. Suppose that $(X, \mathcal{A}), \mu$ is a measure space, that $f\colon X \to Y$ is a function, and let $\mathcal{B} = \{E \subset Y : f^{-1}(E) \in \mathcal{A}\}$. Show that $\nu(E) := \mu(f^{-1}(E))$ defines a measure ν on $\mathcal{B}$.

8. Let X be an uncountable set and let $\mathcal{A}$ be the collection of subsets E of X such that either E or E^c is countable. Define $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E is uncountable. Prove that $\mathcal{A}$ is a σ-algebra and μ is a measure.

9. If X is a Hausdorff topological space find the completion of the Borel sets with respect to a Dirac measure.

10. Show that the 'generalized open intervals' (that is, the 'product of n open intervals') generate the Borel σ-algebra in $\mathbf{R^n}$.

11. Suppose X is the set of real numbers, $\mathcal{B}$ is the Borel σ-algebra, and that μ and ν are two measures on $(X, \mathcal{B})$ such that $\mu((a,b)) = \nu((a,b)) < \infty$ whenever $-\infty < a < b < \infty$. Prove that $\mu(E) = \nu(E)$ whenever $E \in \mathcal{B}$.

12. Let $(X, \mathcal{A}, \mu)$ be a measure space and f a measurable, non-negative function on X. Define $\nu\colon \mathcal{A} \to [0, \infty]$ by $\nu(E) = \int_E f \, d\mu$. Prove that ν is a measure.

3.3 Caratheodory extension theorem: premeasure on an algebra, construction of an outer measure μ^*, the σ-algebra of μ^*-measurable sets. Lebesgue and Lebesgue–Stieltjes measures. Null sets including Cantor set.

1. Show that if μ^* is an outer measure on a set X, then a subset E of X is μ^*-measurable if and only if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ for any sets $A \subset E$ and $B \subset E^c$.

2. If $(X, \mathcal{A}, \mu)$ is a measure space, define $\mu^*(E) = \inf\{\mu(F) : E \subset F, F \in \mathcal{A}\}$ for all subsets E of X. Show that μ^* is an outer measure. Show that each set in $\mathcal{A}$ is μ^*-measurable and μ^* agrees with the measure μ on $\mathcal{A}$.

3. If μ^* is an outer measure on X and $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint measurable sets, then
$$\mu^*\Big(F \cap \Big(\bigcup_{j=1}^{\infty} E_j\Big)\Big) = \sum_{j=1}^{\infty} \mu^*(F \cap E_j) \text{ for any } F \subset X.$$

4. (a) What is Lebesgue outer measure λ^* and Lebesgue measure λ on $\mathbb{R}^n$? Outline the construction (you don't need to prove anything).

   (b) Show that if E is a Lebesgue measurable set, then
$$\begin{aligned}\lambda(E) &= \inf\{\lambda(U) : \text{all open sets } U \text{ containing } E\}\\ &= \sup\{\lambda(K) : K \subseteq E \text{ is compact}\}.\end{aligned}$$

   (c) Prove that Lebesgue measure on $\mathbb{R}^n$ is σ-finite.

   (d) Show that any set of real numbers of positive Lebesgue measure has a subset which is not Lebesgue measurable.

   (e) Let E be a subset of $\mathbb{R}$ which is not Lebesgue measurable. Prove that there exists $\epsilon > 0$ such that $\lambda(B \setminus A) > \epsilon$ for any two Lebesgue measurable sets A, B satisfying $A \subset E \subset B$.

5. Let E be a subset of $\mathbb{R}$ which is not Lebesgue measurable. Prove that there exists an $\alpha > 0$ such that for any two Lebesgue measurable sets A and B satisfying $A \subseteq E \subseteq B$, one has $\lambda(B \setminus A) > \alpha$, where λ denotes Lebesgue measure.

6. Let m be the Lebesgue measure on $\mathbb{R}$. Let $E \subseteq \mathbb{R}$ be Lebesgue measurable such that $0 < m(E) < \infty$. Prove that for all $0 \leq \gamma < 1$ there exists an open interval $I \subseteq R$ such that $m(E \cap I) \geq \gamma m(I)$.

7. Suppose m is Lebesgue measure. Define $x+A = \{x+y : y \in A\}$ and $cA = \{cy : y \in A\}$ for $x \in \mathbb{R}$ and c a real number. Show that if A is a Lebesgue measurable set, then $m(x + A) = m(A)$ and $m(cA) = |c|\, m(A)$.

8. Suppose m is Lebesgue measure and A is a Borel measurable subset of $\mathbb{R}$ with $m(A) > 0$. Prove that if $B = \{x - y : x, y \in A\}$, then B contains a non-empty open interval centered at the origin.

9. Let m be Lebesgue measure on $\mathbb{R}$, and let $A \subset \mathbb{R}$ be a bounded Borel set. Prove that for every $\epsilon > 0$, there exists a set $U \subset \mathbb{R}$ such that U is a finite union of open intervals and $m(U \triangle A) < \epsilon$.

10. Prove the following statements or disprove by giving a counterexample:

    (a) If a set $E \subset \mathbb{R}^n$ has nonempty interior, then $\lambda^*(E) > 0$. (Note: λ^* is the outer measure.)

    (b) If a set $E \subset \mathbb{R}^n$ is such that $\lambda^*(E) > 0$, then E has nonempty interior.

11. Show that if E is any set in $\mathbb{R}^n$, and if $k > 0$, then $\lambda^*(kE) = k^n \lambda^*(E)$.

12. Show that the non-negative scalar multiples of Lebesgue measure are the only translation invariant Borel measures on $\mathbb{R}^n$ which are finite on bounded Borel sets. That is, show that any such measure $= c\lambda$ for a non-negative constant c.

13. Give an example of a nowhere dense subset of $\mathbb{R}$ of positive Lebesgue measure.

14. Show how to adapt the basic (Vitali) construction of a nonmeasurable set to construct a dense set in $[0, 1]$ which is not Lebesgue measurable.

15. Let f be an arbitrary real valued function on $[0, 1]$. Show that the set of points at which f is continuous is a Lebesgue measurable set.

16. Explain the Caratheodory construction/theorem.

17. Exhibit the main properties of the Cantor set and the Cantor ternary function which are typically used in the course (for example, it is a compact set of zero measure).

18. Show that there exists a Lebesgue measurable subset of $\mathbb{R}$ which is not a Borel set. (You don't need to prove anything about the ternary function, just use it.) Explain why Borel sets are Lebesgue measurable.

19. What does it mean to say Lebesgue measure is regular? Prove this in $\mathbb{R}$.

4 Integration

4.1 Measurable functions. Approximation by simple functions.

1. (a) What is a simple (measurable) function?
   (b) Prove that a product of simple (measurable) functions is a simple (measurable) function.
   (c) Show that if f is a real-valued function on $[a, b]$ that is continuous a.e., then f is Lebesgue measurable. Show that it need not be Borel measurable.
2. (a) What is an $\mathcal{A}$-measurable function $f \colon X \to [-\infty, \infty]$? Mention some equivalent conditions.
   (b) What is a Borel measurable function?
   (c) Show how to construct a set which is Lebesgue measurable, but which is not a Borel set.
   (d) Prove that if $f \colon \mathbb{R} \to \mathbb{R}$ is increasing, then f is Borel measurable.
3. For a measure space (X, μ), show that the simple functions are dense in $L^1(X, \mu)$.
4. If $(X, \mathcal{A}, \mu)$ is a measure space and $E \in \mathcal{A}$ show that E is a measure space with respect to a restriction μ_E of μ. If f is a measurable function on $(X, \mathcal{A})$ show that its restriction to E is μ_E-measurable.
5. We always write the decimal expansion of a number so that it does not end in a recurring 9. Show that the function $f_n \colon [0, 1] \to \mathbb{R}$ which takes a number to its nth digit in its decimal expansion, is Borel measurable.
6. Suppose that $f_1, f_2, \cdots$ are measurable functions $X \to [-\infty, \infty]$ which are each finite a.e. Show that for a.e. $x \in X$, $f_n(x)$ is finite for all $n \in \mathbb{N}$.
7. Show that for a function $f \colon X \to \mathcal{C}$, defined on a measurable space $(X, \mathcal{A})$, its real and imaginary parts are both $\mathcal{A}$-measurable, if and only if $f^{-1}(B) \in \mathcal{A}$ for every Borel set B in $\mathcal{C}$.
8. Let $(X, \mathcal{A})$ be a measurable space, and let f_n be a sequence of measurable functions $X \to \mathbb{R}$. Prove that the set of points where $\lim_n f_n(x)$ exists is a measurable set.

4.2 Lebesgue integral: non-negative functions, measurable functions, integrable functions, L^p.

1. (a) What is a simple function? What is an integrable function? How is $\int_X f \, d\mu$ defined? Define it first for a simple function, then for a non-negative measurable function, and then finally for an integrable function.
   (b) Prove that $\int_E f \, d\mu = 0$ if $\mu(E) = 0$ and f is measurable.

(c) What is the connection between f being integrable and $|f|$ being integrable? Prove it. Also, find a function $f\colon [a,b] \to \mathbb{R}$ for which $|f|$ is integrable, but f is not Borel measurable.

2. Prove the following statements or disprove by giving a counterexample:

   (a) Every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is Borel measurable.

   (b) If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at all but one point, then there exists a continuous function g such that $f = g$ a.e.

   (c) Conversely to the last item, if there exists a continuous function g such that $f = g$ a.e. then f is Borel measurable.

   (d) If $A \subset \mathbb{R}$ is an uncountable Lebesgue-measurable set, then its Lebesgue measure is larger than 0.

3. Let $f \in L^1((0,1))$. Define g on $(0,1)$ by

$$g(x) = \int_x^1 \frac{f(t)}{t}\, dt.$$

   Prove that $g \in L^1((0,1))$.

4. Let μ be a finite positive measure on $(X, \mathcal{A})$, let $f \in L^1(X, \mathcal{A}, \mu)$, and let S be a closed set in $\mathbb{F}$. Suppose that the 'averages' $\frac{1}{\mu(E)} \int_E f d\mu$ are in S for every $E \in \mathcal{A}$ with $\mu(E) > 0$. Show that $f(x) \in S$ μ-a.e..

5. Let $f \in L^1(\mathbb{R})$.

   (a) Fix $\alpha > 0$. For $n \in \mathbb{N}$, define f_n by $f_n(x) := \frac{f(nx)}{n^\alpha}$. Show that

$$\|f_n\|_1 = \int_{\mathbb{R}} \frac{|f(nx)|}{n^\alpha} = \int_{\mathbb{R}} \frac{|f(z)|}{n^{1+\alpha}}\, dz = \frac{\|f\|_1}{n^{1+\alpha}}.$$

   (b) Use (a) to show that $f_n(x) \to 0$ as $n \to \infty$ for m-a.e. $x \in \mathbb{R}$ (where m is Lebesgue measure on $\mathbb{R}$.)

6. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are real-valued measurable functions such that $\int_X f\, d\mu = \int_X g\, d\mu$. Prove that either $f = g$ μ-almost everywhere, or there exists a set $E \in \mathcal{A}$ such that $\mu(E) > 0$ and $\int_E f\, d\mu > \int_E g\, d\mu$.

7. Prove, giving all the details, why if $f, g\colon X \to [-\infty, \infty]$ are integrable and $f \le g$ a.e., then $\int_X f \le \int_X g$.

8. Let $(X, \mathcal{A}, \mu)$ be a measure space.

   (a) Prove that if $\mu(X) < \infty$ and if $1 \le p < q < \infty$, then $L^q(\mu) \subseteq L^p(\mu)$.

   (b) Is the statement in (a) true if $\mu(X) = \infty$? If yes, prove it. If no, give a counterexample.

9. Let μ and ν be finite measures on a measurable space $(X, \mathcal{A})$ and suppose that

$$\nu(E) = \int_E f \, d\mu$$

for every $E \in \mathcal{A}$, where f is some function in $L^1(\mu)$. Prove that

$$\int_X g \, d\nu = \int_X gf \, d\mu$$

for all $g \in L^1(\nu)$.

4.3 Monotone convergence theorem, Fatou's lemma, dominated convergence theorem.

1. Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{x^n \cos(x/n)}{(1+x^n)e^x} \, dx$$

exists and compute it.

2. (a) State Lebesgue's dominated convergence theorem.

   (b) Let μ be counting measure on $\mathbb{N}$. We can identify a function $f\colon \mathbb{N} \to \mathbb{R}$ with a sequence $(a_n)_{n\in\mathbb{N}}$. Prove that if f is integrable then $\sum_{k=1}^\infty |a_k|$ converges in the sense of Calculus 2, and the sum equals $\int_\mathbb{N} |f| \, d\mu$. Show also that in this case $\int_X f \, d\mu = \sum_{k=1}^\infty a_k$.

   (c) Show that a Riemann integrable function $f\colon [a, b] \to \mathbb{R}$ is Lebesgue integrable, and that the two integrals of f coincide.

   (d) Complete the sentence: a bounded function $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable if and only if f ....

3. Let λ denote Lebesgue measure on $[0, 1]$. Show that the function $f\colon [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^\infty \frac{\cos(2\pi kx) + x^k}{k^2}$$

is an integrable function and that

$$\int_0^1 f(x) \, d\lambda(x) = \sum_{k=1}^\infty \frac{1}{(k+1)k^2}.$$

4. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence in $L^1([0, 1])$ with $\|f_n\|_1 \le K$ for each $n \in \mathbb{N}$. Prove that if $g\colon [0, 1] \to \mathbb{C}$ and $f_n \to g$ almost everywhere, then $g \in L^1([0, 1])$ and $\|g\|_1 \le K$.

5. Suppose that $\{f_n\}$ is a sequence of nonnegative measurable functions on X that converge a.e. to a function f. If $f_n \le f$ for each n, prove that $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f \, d\mu$. (Note that we are not assuming that f is integrable.)

6. Suppose that $\{f_n\}$ is a sequence of nonnegative measurable functions on X that converge a.e. to a function f. If $\lim_{n\to\infty}\int_X f_n d\mu = \int_X f\ d\mu < \infty$, prove that $\lim_{n\to\infty}\int_E f_n\, d\mu = \int_E f\ d\mu$ for any measurable set E. [Hint: Fatou on E and on E^c.]

7. Show that the function $f(x) = \frac{1}{\sqrt{x}}$ is Lebesgue integrable on $[0,1]$, but $1/x$ is not.

8. Show that the monotone and dominated convergence theorems both fail for the Riemann integral, by finding an increasing sequence of nonnegative bounded Riemann integrable functions on $[0,1]$, which converge everywhere to a bounded function f, but f is not Riemann integrable.

9. Show that a Riemann integrable function is not necessarily Borel measurable.

10. Suppose that E_n are disjoint sets in $\mathcal{A}$ for $n \in \mathbb{N}$, with $X = \cup_n E_n$ and that f is a measurable nonnegative or integrable function on X. Show that $\int_X f = \sum_n \int_{E_n} f$, and show the latter series converges absolutely if f is integrable.

4.4 Almost everywhere equality, convergence. Other types of convergence: in measure, in L^p, uniform, in L^∞. Implications and counterexamples.

1. Suppose that a series $\sum_{n=1}^{\infty} f_n$ converges absolutely in $L^1(\mathbb{R})$, i.e. $\sum_{n=1}^{\infty}\|f_n\|_{L^1} < \infty$. Prove that:

   (a) the series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for a.e. $x \in \mathbb{R}$;

   (b) $f \in L^1(\mathbb{R})$;

   (c) $\int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n$.

2. For a real-valued function f on $\mathbb{R}$, define the translate f^t by $f^t(x) = f(x-t)$.

   (a) Suppose f is continuous on $\mathbb{R}$ and has compact support. Prove that $\left\|f^t - f\right\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to 0$.

   (b) Show that if $f \in L^1(\mathbb{R})$, then $\left\|f^t - f\right\|_{L^1(\mathbb{R})} \to 0$ as $t \to 0$.

3. Prove the following statements or disprove by providing a counterexample:

   (a) If f_k is a sequence of non-negative and measurable functions on $[0,1]$ such that $f_k \to 0$ a.e., then $\int_{[0,1]} f_k \to 0$.

   (b) If f_k is a sequence of non-negative and measurable functions on $[0,1]$ such that $f_k \le M$ for all k and $f_k \to 0$ a.e., then $\int_{[0,1]} f_k \to 0$.

   (c) If f_k is a sequence of non-negative and measurable functions on $[0,1]$ such that $f_k \to 0$ in measure, then $f_k \to 0$ a.e.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n)_{n=1}^{\infty}$ be a sequence of μ-integrable functions and suppose that f is μ-integrable as well.

(a) Prove that if $f_f \to f$ in the $L^1(\mu)$ sense (or even in L^p), then $f_n \to f$ in measure.

(b) If $\mu(X) < \infty$ and if $f_n \to f$ in measure, does it follow that $f_n \to f$ in the $L^1(\mu)$ sense? Either prove this or give a counterexample.

5. Let $X = [0,1]$ with Lebesgue measure.

   (a) Let $f_n(x) = \cos(2\pi nx)$. Then $f_n \to 0$ weakly in $L^2(X)$ but $f_n \not\to 0$ a.e. or in measure.

   (b) Let $f_n(x) = n\chi_{(0,1/n)}$. Then $f_n \to 0$ a.e. and in measure, but $f_n \not\to 0$ weakly in L^p for any p.

6. Let $1 \le p < \infty$ and suppose that $f_n, f \in L^p(\mathbb{R}^n)$ satisfy $f_n \to f$ a.e. Prove that

$$\|f_n - f\|_{L^p} \to 0 \iff \|f_n\|_{L^p} \to \|f\|_{L^p}.$$

7. Let $E \subset \mathbb{R}^n$ be a measurable set and f an integrable function on E.

   (a) Set $E_m = \{|f| < m\}$ and show that $f\chi_{E_m} \to f$ in L^1-norm as $m \to \infty$.

   (b) Given $\epsilon > 0$, show that there exists a constant $\delta > 0$ such that for every measurable set $A \subset E$, we have $\lambda(A) < \delta \implies \int_A |f| < \epsilon$.

   (c) Show that, given $\epsilon > 0$, there exists a measurable set $A \subset E$ such that f is bounded on A and $\int_{E\setminus A} |f| < \epsilon$.

8. Fix $1 \le p < \infty$ and suppose that:

   (i) $f_m \to f$ in measure;

   (ii) for each $\epsilon > 0$, there exists a $\delta > 0$ such that for every measurable set $E \subset \mathbb{R}^n$ satisfying $\lambda(E) < \delta$, we have $\int_E |f_m|^p < \epsilon$ for every m;

   (iii) for each $\epsilon > 0$, there exists a measurable set $E \subset \mathbb{R}^n$ such that $\lambda(E) < \infty$ and $\int_{E^c} |f_m|^p < \epsilon$ for every m.

   Prove that $f_m \to f$ in $L^p(\mathbb{R}^n)$.

9. Let $(X, \mathcal{A}, \mu)$ be a measure space.

   (a) Let $f\colon X \to [0,\infty]$ be measurable and suppose that $\int_X f\,d\mu = 0$. Prove that $f = 0$ a.e.

   (b) Let $f \in L^1(X,\mu)$. Show that if $\int_A f\,d\mu = 0$ for every $A \in \mathcal{A}$, then $f = 0$ a.e.

10. Construct a sequence of bounded continuous functions on $[0,1]$ whose integrals are all zero, but such that the sequence converges at no point.

11. Let $k \in L^1(\mathbb{R})$ be such that $\int_R k(x)\,dx = 1$ and, for each $n \in N$, set $k_n(x) = nk(nx)$. Prove that for every $f \in L^1(\mathbb{R})$,

$$\lim_{n\to\infty} \|f * k_n - f\|_{L^1} = 0.$$

   Recall that convolution is defined by $f * k(x) = \int_{\mathbb{R}} f(x-y)k(y)\,dy$.

12. Let $F \subset \mathbb{R}$ be a closed subset of positive measure. For $x \in \mathbb{R}$, define the distance from x to F by

$$d(x, F) := \inf_{z \in F} d(x, z).$$

Prove that for Lebesgue almost every $y \in F$, we have

$$\lim_{x \to y} \frac{d(x, F)}{|x - y|} = 0.$$

Hint: Consider Lebesgue density points of F.

13. (a) Prove that if $f \in L^p([0,1])$ and if $2 < p < \infty$, then the integral

$$\int_0^1 \frac{|f(x)|}{\sqrt{x}} \, dm(x)$$

is finite.

(b) Prove or give counterexample: if $f \in L^2([0,1])$, then the integral in part (a) is finite.

4.5 Egorov and Luzin theorems.

1. Show that under the conditions of the dominated convergence theorem one can deduce that $f_n \to f$ almost uniformly. [Hint: use the proof of Egoroff's theorem, but at one point one needs to appeal to LDCT and another result.]

2. Show using Lusin's theorem that any integrable (resp. bounded measurable) function on $[a, b]$ is a.e. an almost uniform limit of a sequence (resp. bounded sequence) of continuous functions. Show a bounded function on $[a, b]$ is Lebesgue measurable if and only if it is a.e. equal to a pointwise limit of continuous functions.

4.6 Monotone class theorem, product σ-algebra, product measure, Fubini–Tonelli theorem

1. What is a monotone class? State and prove the monotone class theorem.

2. Find an example of a set X and a monotone class $\mathcal{M}$ consisting of subsets of X such that $\emptyset \in \mathcal{M}$, $X \in \mathcal{M}$, but $\mathcal{M}$ is not a σ-algebra.

3. Give precise statements (e.g., as stated in class or in Folland's book) of Tonelli's theorem and Fubini's theorem for general measure spaces.

4. (a) If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces, define the σ-algebra $\mathcal{A} \times \mathcal{B}$. Also explain briefly how the product measure $\mu \times \nu$ is defined.

   (b) What are the relationships between $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}^2)$, $\mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$, and $\mathcal{L}(\mathbb{R}^2)$? Prove most of your assertions.

   (c) State and prove Tonelli's theorem for $\mu \times \nu$.

(d) State and prove Fubini's theorem for $\mu \times \nu$.

5. Let λ denote Lebesgue measure on $\mathbb{R}$, and let $\lambda^2 := \lambda \times \lambda$ denote the product measure on $\mathbb{R}^2$. Prove that if $f, g \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}^2} f(x)g(t-x)\, d(\lambda^2)(x,t) = \int_{\mathbb{R}^2} g(x)f(t-x)\, d(\lambda^2)(x,t).$$

Justify any steps in your proof, and if you use any theorems explain why the hypotheses of those theorems are satisfied.

6. Let $f \in L^1((0,1]^2, \lambda_2)$ such that $\int_{(0,x]\times(0,y]} f\, d\lambda_2 = 0$ for every $x, y \in (0,1]$. Prove that $f = 0$ λ_2-a.e.

7. Give an example of a $\mathcal{L}(\mathbb{R}^2)$-measurable function which is not $\mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$-measurable.

8. If $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ for $x, y \in (0,1]$, show that $\int_0^1 \int_0^1 f\, dx\, dy \neq \int_0^1 \int_0^1 f\, dy\, dx$. What is going wrong here in Fubini's theorem?

9. Show that Lebesgue measure and the Lebesgue integral are invariant under rotations.

10. Prove that one may interchange absolutely convergent double sums.

11. Suppose that E is a dense set in $\mathbb{R}$, and f is a real-valued function on $\mathbb{R}^2$ such that (i) each section f_x is Lebesgue measurable for each $x \in E$ and (ii) each section f^y is continuous for a.e $y \in \mathbb{R}$. Prove that f is Lebesgue measurable on $\mathbb{R}^2$.

12. Let h and g be integrable functions on X and Y respectively, and let $f(x,y) = h(x)g(y)$. Show that f is integrable and

$$\int f\, d(\mu \times \nu) = \left(\int_X h\, d\mu\right)\left(\int_Y g\, d\nu\right).$$

State and prove the corresponding theorem if h, g are measurable and non-negative.

13. What are the relationships between $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}^2)$, $\mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$, and $\mathcal{L}(\mathbb{R}^2)$? Prove these.

14. If $(X_j, \mathcal{A}_j)$ is a measurable space for $j = 1, 2, 3$, then $\bigotimes_{j=1}^{3} \mathcal{A}_j = (\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes \mathcal{A}_3$. Moreover, if μ_j is a σ-finite measure on $(X_j, \mathcal{A}_j)$, then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

15. True or false? Prove or give a counter example.

    (a) Let $E \subseteq \mathbb{R}$ be a Borel set, then the set $\{(x,y) \in \mathbb{R}^2 : x - y \in E\}$ is a Borel set in $\mathbb{R}^2$.

    (b) Let $E \subseteq [0,1] \times [0,1]$. Assume that for every $x, y \in [0,1]$ the sets $E_x = \{y \in [0,1] : (x,y) \in E\}$ and $E^y = \{x \in [0,1] : (x,y) \in E\}$ are Borel. Then E is Borel.

16. If $f\colon X \to [0,\infty]$, define $E = \{(x,y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\}$. This is 'the region under the graph of f'.

(a) Prove that if f is measurable, then E is in $\mathcal{A} \times \mathcal{L}(\mathbb{R})$.

(b) Prove that under the hypothesis of (a), $(\mu \times \lambda)(E) = \int_X f \, d\mu$. Note that some books define the integral by this formula.

17. Suppose $f \colon X \times Y \to [0, \infty]$ is measurable with respect to $\mathcal{A} \times \mathcal{B}$, and for μ-a.e. x in X we know that $f(x, y)$ is finite ν-a.e. Prove for ν-a.e. y in Y that $f(x, y)$ is finite μ-a.e.

4.7 Signed measure, positive and negative sets. Hahn and Jordan decompositions, total variation measure, complex measures.

1. What is a complex measure ν on $(X, \mathcal{A})$?

2. Define the variation $|\nu|$. Also, complete the sentence: "The variation $|\nu|$ is the smallest positive measure such that ..." Define $\|\nu\|$.

3. Prove that $|\nu + \sigma| \leq |\nu| + |\sigma|$ and $|c\nu| = |c||\nu|$ for complex measures ν, σ on $(X, \mathcal{A})$.

4. If ν is a signed measure, E is ν-null if and only if $|\nu|(E) = 0$. Also if ν and μ are signed measures, $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

5. If ν is a complex measure on $(X, \mathcal{A})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

6. Let ν be a signed measure on $(X, \mathcal{A})$.

   (a) $L^1(\nu) = L^1(|\nu|)$.

   (b) If $f \in L^1(\nu)$, $\left|\int f \, d\nu\right| \leq \int |f| \, d|\nu|$.

   (c) If $E \in \mathcal{A}$, $|\nu|(E) = \sup\{\left|\int_E f \, d\nu\right| : |f| \leq 1\}$.

7. If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

4.8 Mutually singular measures, absolutely continuous measures, Cantor–Lebesgue function.

1. If ν and μ are complex measured and λ is a positive measure, then $\nu \perp \mu$ if and only if $|\nu| \perp |\mu|$ and $\nu \ll \lambda$ if and only if $|\nu| \ll \lambda$.

5 Differentiation

1. Suppose that $f \colon [0, 1]^2 \to \mathbb{R}$ satisfies the following conditions:

   (i) For each fixed $x \in [0, 1]$, $f(x, y)$ is an integrable function of y;

   (ii) $\frac{\partial f}{\partial x}(x, y)$ exists at all points and is bounded on $[0, 1]^2$.

Prove that $\frac{\partial f}{\partial x}(x, y)$ is a measurable function of y for each $x \in [0, 1]$ and that

$$\frac{d}{dx}\int_0^1 f(x, y)\, dy = \int_0^1 \frac{\partial f}{\partial x}(x, y)\, dy.$$

2. Suppose $f\colon [0, 1] \to \mathbb{R}$ and $f(0) = 0$. For each of the following statements about f, indicate whether the statement is TRUE or FALSE. If FALSE, provide a counterexample. (For every counterexample, you can simply describe the function and state its properties; you don't need to prove that it has those properties or go into great detail.)

   (a) If there exists $g \in L^1([0, 1])$ such that $f(x) = \int_0^x g(t)\, dm(t)$ for all $x \in [0, 1]$, then f is differentiable at almost every $x \in [0, 1]$.

   (b) If f is differentiable at almost every $x \in [0, 1]$ and $f'(x) = 0$ whenever f is differentiable at $x \in [0, 1]$, then $f(1) = 0$.

   (c) If f is absolutely continuous, and $f'(x) = 0$ whenever f is differentiable at $x \in [0, 1]$, then $f(1) = 0$.

3. Show that if a scalar valued function F on $\mathbb{R}$ or $[a, b]$ is differentiable a.e., then F' is Lebesgue measurable.

4. Find a strictly increasing continuous function f on $\mathbb{R}$, with $f'(x) = 0$ a.e..

5. Let m denote Lebesgue measure on $\mathbb{R}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function for which there exists $C > 0$ such that $\left|F(x)\right| \leq C|x|$ for every $x \in \mathbb{R}$. Further suppose that F is differentiable at 0 with $F'(0) = a$. Prove that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \frac{nF(x)}{x(1 + n^2x^2)}\, dm(x) = \pi a.$$

Hint: Consider the change of variable $u = nx$. You may use the fact that

$$\int_{-\infty}^{\infty} \frac{1}{1 + u^2}\, dm(u) = \pi.$$

5.1 Radon–Nikodym theorem, Lebesgue decomposition theorem.

1. (a) Find an example to show that the hypothesis in the Radon-Nikodym theorem that μ is σ-finite cannot be omitted.

   (b) Show that the Lebesgue decomposition theorem is not true for arbitrary positive measures.

2. Show that if $\nu \in M(X, \mathcal{A})$, and if μ is a positive measure on $\mathcal{A}$, then $\nu \ll \mu$ if and only if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $E \in \mathcal{A}$ and $\mu(E) < \delta$. If you like, assume that the measures are σ-finite.

3. Find an example to show that the hypothesis in the Radon-Nikodym theorem that μ is σ-finite cannot simply be removed.

4. Let $\mathcal{L}$ denote the Lebesgue measurable subsets of $\mathbb{R}$ and let m denote Lebesgue measure on $(\mathbb{R}, \mathcal{L})$. Define signed measures μ and ν on $(\mathbb{R}, \mathcal{L})$ by

$$\mu(E) := \int_E |x|\; dm(x) \text{ and } \nu(E) := \int_{E \cap [-1,\infty)} x\, dm(x).$$

   (a) Prove that $\nu \ll \mu$ and find $\frac{d\nu}{d\mu}$.

   (b) Either prove or disprove that $\mu \ll |\nu|$. (The symbol $|\nu|$ denotes the total variation measure of ν.)

5. For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{A}_j)$ such that $\nu_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\nu_2}(x_2).$$

6. Let $X = [0, 1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure, and $\mu =$ counting measure on $\mathcal{A}$.

   (a) $m \ll \mu$ but $dm \neq f\, d\mu$ for any f.

   (b) μ has no Lebesgue decomposition with respect to m.

5.2 Locally integrable functions on $\mathbb{R}^n$ are equal to local averages Lebesgue-a.e.: Vitali covering lemma, Lebesgue density theorem.

1. If $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, where $1 \leq p < \infty$, show that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, so that a.e. $\vec{x} \in \mathbb{R}^n$ is a Lebesgue point of f. Also show that in this case $\frac{1}{\lambda(B(\vec{x},r))} \int_{B(\vec{x},r)} |f(\vec{y}) - f(\vec{x})|^p\, d\vec{y} \to 0$ as $r \to 0^+$, for a.e. $\vec{x} \in \mathbb{R}^n$.

5.3 Bounded variation, Lipschitz continuity, absolutely continuous functions. BV functions are the difference of increasing functions, hence a.e.-differentiable. Fundamental theorem of calculus for AC functions.

1. Let m denote Lebesgue measure on $\mathbb{R}$.

   (a) Let $a < b$ be real numbers. Give the definition of an absolutely continuous function $f\colon [a, b] \to \mathbb{R}$.

   (b) Show that every $F \in AC([a, b])$ is of bounded variation.

   (c) Prove that if $F \in AC([a, b])$ then the length of the graph of F equals $\int_a^b \|F'(t)\|\, dt$.

   (d) Suppose $f\colon [a, b] \to \mathbb{R}$. Prove that if A is a Lebesgue measurable subset of $[a, b]$ with $m(A) = 0$, then $m(f(A)) = 0$.

(e) if A is a Lebesgue measurable subset of $\mathbb{R}$ with $m(A) = 0$, does it follow that

$$\{e^x : x \in A\}$$

has Lebesgue measure zero? Either prove this or give a counterexample?

2. (a) A function $f\colon [0,1] \to \mathbb{R}$ is said to be *Lipschitz* if there exists $L > 0$ such that $\left|f(x) - f(y)\right| \leq L|x - y|$ for all $x, y \in [0,1]$. Prove that if $f\colon [0,1] \to R$ is Lipschitz, then f is absolutely continuous.
   (b) Suppose $f\colon [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and absolutely continuous on $[\epsilon, 1]$ for every $\epsilon \in (0,1)$. Show that f need not be absolutely continuous on $[0,1]$ by giving a counterexample. Hint: Consider functions of the form $x^\alpha \cos(1/x^b)$.
3. (a) Define the terms: AC (absolute continuity), NBV.
   (b) Explain the fundamental correspondence between measures on $\mathbb{R}$ and NBV, and between $L^1(\mathbb{R})$ and AC.
   (c) Show that every $f \in AC[a, b]$ is of bounded variation.
4. (a) What does it mean that measurable sets E_n 'shrink nicely' to x?
   (b) Complete the sentence: "If $f \in L^1(\mathbb{R}^k)$ and if x is a Lebesgue point of f and E_n shrink nicely to x, then …"
   (c) State and prove the 'first fundamental theorem of calculus' involving NAC.
   (d) Prove that a function on $\mathbb{R}$ is Lipschitz iff it is in AC and its derivative is in L^∞.
5. State and prove the 'second fundamental theorem of calculus' involving AC($[a, b]$).
6. Show that every $f \in$ AC($[a, b]$) is of bounded variation.
7. Show that a function F on the reals is Lipschitz continuous if and only if F is absolutely continuous and F' is essentially bounded.
8. Show that a function $F\colon (a, b) \to \mathbb{R}$ is convex if and only if F is absolutely continuous on every subinterval $[c, d]$ of (a, b) and F' is increasing on the set on which it is defined.
9. State and prove a form of Jensen's inequality.

6 Basics of functional and Fourier analysis

6.1 L^p spaces: conjugate exponents, basic inequalities (Young, Hölder, Minkoswki). Density of simple functions. Convolution, mollification, density of C_c^∞ in L^p.

1. State Hölder's inequality and Minkoswki's inequality.
2. When does equality hold in Minkoswki's inequality?

3. Show that continuous functions on $[a, b]$ are dense in $L^p([a, b])$ if $1 \leq p < \infty$.

4. Show that $L^p([a, b])$ is separable if $1 \leq p < \infty$.

5. Define the convolution $f * g$ of two functions in $L^1(\mathbb{R})$. Show that the function inside the integral in the definition of $f * g$ is measurable. Show also that $f * g = g * f$. Lastly, prove that
$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 .$$
You must prove this special case of the Young inequality yourself; do not simply quote the Young inequality.

6. Show that convolution is associative, and distributive on $L^1(\mathbb{R}^n)$.

7. Fix $1 < p < \infty$ and let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^p([0, 1])$ for which there exists $K > 0$ such that $\|f_n\|_p \leq K$ for every $n \in \mathbb{N}$. Suppose that there exists a Lebesgue measurable function f on $[0, 1]$ such that $f_n(x) \to f(x)$ for m-a.e. $x \in [0, 1]$.

   (a) Prove that $f \in L^p([0, 1])$ and $\|f\|_p \leq K$.

   (b) Prove that for every $g \in L^q([0, 1])$, we have
   $$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)\, dx = \int_0^1 f(x)g(x)\, dx.$$

   (c) Is the statement in part (b) true if $p = 1$ and $q = \infty$? If yes, prove it. If no, give a counter example.

8. Show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $(f * g)(x)$ exists for all x, and $f * g$ is bounded with $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$.

6.2 Normed vector spaces, dual space, duals of L^p spaces. Hahn–Banach theorem and its consequences, isometric embedding $X \to X^{**}$

1. Use the Hahn-Banach theorem to show that the canonical map from a normed space Z into Z^{**} is an isometry.

2. If X, Y are normed vector spaces, the map $(T, x) \mapsto Tx$ is continuous from $L(X, Y) \times X$ to Y. (That is, if $T_n \to T$ and $x_n \to x$, then $T_n x_n \to Tx$.)

3. Prove that $\ell_2^1 \cong \ell_2^\infty$ isometrically. This is over the real number field.

4. Suppose that X is a finite-dimensional vector space. Let $e_1, \ldots, e_n$ be a basis for X and define $\left\| \sum_{j=1}^n a_j e_j \right\|_1 = \sum_{j=1}^n |a_j|$.

   (a) $\|\cdot\|_1$ is a norm on X.

   (b) the map $(a_1, \ldots, a_n) \mapsto \sum_{j=1}^n a_j e_j$ is continuous from $\mathbb{C}^n$ with the usual Euclidean topology to X with the topology defined by $\|\cdot\|_1$.

(c) $\{x \in X : \|x\|_1 = 1\}$ is compact in the topology defined by $\|\cdot\|_1$.

(d) All norms on X are equivalent.

5. Let $(X, \mathcal{A})$ be a measurable space and let $M(X)$ be the space of all complex measures on $(X, \mathcal{A})$. Then $\|\mu\| := |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space.

6. Suppose that X and Y are Banach spaces.

   (a) If $\{T_n\}_{n=1}^{\infty} \subset L(X, Y)$ and $T_n \to T$ weakly (or strongly), then $\sup_n \|T_n\| < \infty$.

   (b) Every weakly convergent sequence in X, and every weak* convergent sequence in X^*, is bounded (with respect to the norm).

7. Let X be a normed space, and let $\tilde{X}$ be its completion. Show that for any Banach space Y, we have $B(X, Y) \cong B(\tilde{X}, Y)$ isometrically. Show also that if X and Y are linear subspaces of Banach spaces, and if $T\colon X \to Y$ is an isometry, then T extends to a unique isometry $T\colon \bar{X} \to \bar{Y}$ whose range is closed.

8. If X is a separable normed linear space, the weak* topology on the closed unit ball in X^* is second countable and hence metrizable.

9. A vector subspace of a normed vector space X is norm-closed if and only if it is weakly closed.

6.3 Fourier transform on L^1, properties, Riemann–Lebesgue lemma, Schwartz class, Fourier inversion formula, Plancherel's theorem, extension of Fourier transform to L^2. Fourier series using Hilbert spaces.

1. Define the Fourier transform $\mathcal{F}(f) = \hat{f}$ for $f \in L^1(\mathbb{R}^n)$, and show that it is well defined and in $C_b(\mathbb{R}^n)$.

2. State and prove a formula for $\mathcal{F}(f')$ if $f, f' \in C_0(\mathbb{R})$.

3. Show that the Fourier transform of $f * g$ is a product of the Fourier transform of f and the Fourier transform of g.

4. Let $\mathcal{F}[.]$ denote the Fourier transform. Prove that there does not exist $u \in L^1(\mathbb{R})$ such that $f = f * u$ a.e. for every $f \in L^1(\mathbb{R})$. Hint: proceed by contradiction. Assume u exists and let $f \in L^1(\mathbb{R})$ be a function such that $\mathcal{F}[f](\gamma) \neq 0$ for all $\gamma \in \mathbb{R}$ (you do not need to give an example of such an f). Now use

$$\left\|\mathcal{F}[f - f * u]\right\|_{L^\infty(\mathbb{R})} \leq \|f - f * u\|_{L^1(\mathbb{R})} = 0$$

   to deduce a contradiction.

5. State a form of the Fourier inversion theorem.

6. (i) If $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$, show that (e_n) is an orthonormal set in $L^2([0, 2\pi])$.
   (ii) Define the Fourier series of a function $f \in L^2([0, 2\pi])$.
   (iii) Show that for every $f \in L^2([0, 2\pi])$, the Fourier series of f converges to f in L^2-norm. (As part of this you will show that (e_n) above is an orthonormal basis.)

6.4 Banach spaces: L^p, ℓ^p, C, C^r, C^α, $L(X, Y)$. Separability. Baire category theorem and its consequences: uniform boundedness principle (Banach–Steinhaus theorem), open mapping theorem, closed graph theorem.

1. (a) Consider a compact Hausdorff space K, and let $C(K)$ be the continuous scalar functions on K. Define the supremum norm on $C(K)$, show that it is a norm, and that with this norm $C(K)$ is a normed space. If you are lost, do this in the case $K = [0, 1]$.
   (b) With notation in (a), explain carefully the relationship between the measures on K and linear functionals on $C(K)$. Consider two cases, the positive (or nonnegative) case and the general case.
   (c) Show that with this norm $C(K)$ is a Banach space.
   (d) What can you say about a surjective linear continuous one-to-one function between Banach spaces? Also state the principle of uniform boundedness and prove it.
   (e) Define the terms "dense" and "separable". Give a simple test for when a normed space is not separable, and using this, show that $L^\infty([0, 1])$ is not separable.

2. Define $L^p(X, \mu)$ for $1 \leq p \leq \infty$. Prove in full detail that $L^1(X, \mu)$ is a normed space and a Banach space.

3. Show that $L^p(X, \mu)$ is reflexive for $1 < p < \infty$.

4. Show that $\ell^p \subseteq \ell^q$ if $p \leq q$, and prove that $\|\cdot\|_q \leq \|\cdot\|_p$ in this case.

5. Prove that $(\ell^1)^*$ and ℓ^∞ are isometrically isomorphic.

6. Show that ℓ^1 is separable but not reflexive.

7. (a) State the closed graph theorem.
   (b) Let X and Y be (complex) Banach spaces. Prove that if $T\colon X \to Y$ is a linear map such that $\phi \circ T \in X^*$ for every $\phi \in Y^*$, then T is a bounded linear map. Here X^* and Y^* denote the dual spaces of X and Y, respectively.

8. Let m denote Lebesgue measure on $[0, 1]$. Let (f_n) be a sequence of functions in $L^2([0, 1])$ that converges weakly to $f \in L^2([0, 1])$, meaning that

$$\lim_{n\to\infty} \int_0^1 f_n g \, dm = \int_0^1 f g \, dm$$

for every $g \in L^2([0,1])$. Prove that there exists $K > 0$ such that $\|f_n\|_{L^2([0,1])} \leq K < \infty$ for every $n \in \mathbb{N}$. Hint: Uniform boundedness principle.

6.5 Riesz representation theorem for $C(X)^*$. Regularity of measures.

1. Suppose F is a closed subspace of $[0,1]$ and define

$$L(f) = \int_0^1 f\chi_F \, dx$$

for real-valued continuous functions f on $[0,1]$. Prove that if μ is the measure whose existence is given by the Riesz representation theorem, then $\mu(A) = m(A \cap F)$, where m is Lebesgue measure.

2. Suppose X and Y are compact metric spaces and $F\colon X \to Y$ is a continuous map from X onto Y. If ν is a finite measure on the Borel sets of Y, prove that there exists a measure μ on the Borel sets of X such that

$$\int_Y f \, d\nu = \int_X f \circ F \, d\mu$$

for all f that are continuous on Y.

3. Prove that if X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra, and μ, ν are two finite positive measures (two finite signed measures) on $(X, \mathcal{B})$ such that

$$\int f \, d\mu = \int f \, d\nu$$

for all $f \in C(X)$, then $\mu = \nu$.

4. Prove that if X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra, and μ is a complex measure on $(X, \mathcal{B})$, then the total variation of μ equals

$$\sup_{f \in C(X),\, \sup|f| \leq 1} \left| \int f \, d\mu \right|.$$

5. Prove that if you apply the Riesz-Markov-Kakatani theorem to the linear functional which is the *Riemann* integral on $C([0,1])$, then one obtains Lebesgue measure on $[0,1]$. What is the name of the σ-algebra here?

6.6 Hilbert spaces: inner product, Cauchy–Schwarz, parallelogram law, polarization identity, orthogonal complements and projections, Riesz representation (relate H and H^*), orthonormal sets, Gram–Schmidt procedure, Bessel's inequality, completeness, Parseval's identity, orthonormal basis.

1. Prove that $L^2(X, \mu)$ is a Hilbert space.

2. Let $\mathcal{H}$ and $\mathcal{H}'$ be two Hilbert spaces. Prove that a linear map from $\mathcal{H}$ to $\mathcal{H}'$ is unitary if and only if it is isometric and surjective. Also characterize unitaries in terms of an orthonormal basis.

3. Prove that up to unitary, there is a unique infinite dimensional separable Hilbert space.

4. Prove that if M is a closed subspace of a Hilbert space $\mathcal{H}$, then $(M^{\perp})^{\perp} = M$. Is this necessarily true of M is not closed? If not, give a counter example.

5. Prove that if $\mathcal{H}$ is infinite dimensional, then the closed unit ball in $\mathcal{H}$ is not compact.

6. Show that Hilbert spaces are reflexive.

7. If M is a closed subspace of a Hilbert space $\mathcal{H}$, let $x + M = \{x + y : y \in M\}$.

   (a) Prove that $x + M$ is a closed convex subset of $\mathcal{H}$.

   (b) Let Qx be the point of $x + M$ of smallest norm and $Px = x - Qx$. P is called the *projection* of x onto M. Prove that P and Q are both mappings of $\mathcal{H}$ into M and $M^{\perp}$, respectively.

   (c) Prove that P and Q are linear mappings.

   (d) Prove that if $x \in M$, $Px = x$ and $Qx = 0$.

   (e) Prove that if $x \in M^{\perp}$, $Px = 0$ and $Qx = x$.

   (f) Prove that $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.

8. Show that every orthonormal set in a Hilbert space H is contained in an orthonormal basis for H. Show also that every Hilbert space has an orthonormal basis.

9. Prove that every orthonormal sequence in a Hilbert space converges weakly to 0.

10. Use Parseval's identity with the function $f(x) = x$ on $[0, 2\pi)$ to derive the formula

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$