Complex Analysis Preliminary Exam, January 13, 2016

Name: ______________

There are two parts of the exam. On this side are true-false questions and on the reverse side you will find problems requiring proofs.

True-False problems Circle either "T" if the statement is true, or "F" if the statement is false.

1. Let $a \in \mathbb{C}$ be a point and it is an isolated singularity of a rational function f. Then a could be an essential singularity for f.

2. Let $\Omega = \Delta(0,1) - \{0\}$. Any function $f \in Hol(\Omega)$ is the derivative of some other function $g \in Hol(\Omega)$.

3. There is no holomorphic function f defined on the punctured disk $\Delta(1) - \{0\}$ such that f' has a simple pole at 0.

4. Let $\Delta(1)^+ = \{z \in \Delta(1) \mid Im(z) > 0\}$ be the upper half disk in $\mathbb{C}$. Let f be a holomorphic function defined in $\Delta(1)^+$ and it is continuous to the closure $\overline{\Delta(1)^+}$. Then by using Schwarz Reflection Principle, f can be extended holomorphically in $\Delta(1)$.

5. By Riemann mapping theorem, any simply connected domain $\subseteq \mathbb{C}$ can be mapped by a biholomorphic map onto the unit disk.

Problems required proofs

1. Complete the following steps:

   (a) State Liouville's theorem

   (b) Computing $\oint_{\partial\Delta(0,R)} \frac{f(z)}{(z-a)(z-b)} dz$ where f is an entire function and $a, b \in \mathbb{C}$ with $a \neq b$.

   (c) Prove Liouville's theorem from (a) and (b) above.

2. Let f be holomorphic in the unit disk $\Delta(1)$ and continues on $\overline{\Delta(1)}$. Assume that

$$|f(z)| = |e^z| \quad \forall z \in \partial\Delta(1).$$

   Find all such f.

3. Let f be holomorphic function defined on the unit disk $\Delta(1)$ with the radius of convergence 1. Prove that there is at least one point in the boundary $\partial\Delta(1)$ at which the function f cannot extend holomorphically.

4. Evaluate the real integral

$$\int_0^\infty \frac{log\ x}{1+x^4} dx.$$

5. Show that the polynomial $2z^5 - 6z^2 + z + 1$ has exactly three zeros (counting multiplicities) in $\{z \mid 1 < |z| < 2\}$.

6. Prove the Schwatz-Pick lemma: Let $f : \Delta(1) \to \Delta(1)$ be holomorphic. Then

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leq \left|\frac{z-a}{1-\overline{a}z}\right|, \quad \forall a, z \in \Delta(1).$$