Applicable Analysis Preliminary Exam August 2009

Your grade will be based on your answers to the first three questions and 3 of the last five questions. In question 1, each correct answer is worth 4 points, in question 2 each correct answer is worth 5 points and in question 3 a correct answer is worth 4 points, an incorrect answer deducts 2 points while no answer is 0 points. The remaining questions are worth either 13, or 14, points each.

The notation is that used in the courses given in 2008-9. All vector spaces are real vector spaces. H_1, H_2 will be real Hilbert spaces with inner products $\langle ., . \rangle_j$ j = 1 or 2. Ask the exam supervisor if you have other questions about the notation.

Question 1: (4 points each)

(a): Let A be a real $m \times n$ matrix. Define the rank of A.

(b): Suppose X is a normed vector space and $f : X \to X$ is a function. What does it mean to say that f is Lipschitz continuous on X?

(c): Let H be a Hilbert space and $\mathcal{E}$ be a subset of H. What does it mean to say that $\mathcal{E}$ is an orthonormal basis of H.

(d): Let H be a Hilbert space, and $L : H \to H$ be a continuous linear transformation. Define the operator norm of L.

(e): Suppose $L : H_1 \to H_2$ is a continuous linear transformation. Define the adjoint operator L^*. What is its domain and range?

Question 2: (5 points each)

State carefully the following results; making sure that all conditions are included and significant terms are defined.

(a): The Weierstrass existence theorem for minimizers of a function $f : K \to \mathbb{R}$ with K a subset of $\mathbb{R}^n$.

(b): The local inverse function theorem for a function $f : I \to \mathbb{R}^n$ where I is an nonempty open subset of $\mathbb{R}^n$.

(c): The Fredholm splitting theorem for a continuous linear transformation $L : H_1 \to H_2$.

(d): The Lax-Milgram theorem.

Question 3: (4 points for a correct answer, -2 points for an incorrect answer and 0 points for no answer). Answer T (true) or F (false) for each of the following statements.

(a) If K is an open convex set in $\mathbb{R}^n$, $f : K \to \mathbb{R}$ is convex and $\nabla f(\hat{x}) = 0$ then $\hat{x}$ minimizes f on K.

(b) If $b \in \mathbb{R}^m$, A is a real $m \times n$ matrix and there is a $c > 0$ such that

$$\|Ax\|_2 \geq c\,\|x\|_2 \qquad \text{for all } x \in \mathbb{R}^n$$

then there is a unique solution of $Ax = b$.

(c) If $L : H \to H$ be a continuous linear transformation and L is one-to-one (injective), then L* is one-to-one.

(d) If K is an closed convex set in $\mathbb{R}^n$, $f : K \to \mathbb{R}$ is continuous and convex, then $f(x)^2$ is continuous and convex on K.

(e) Suppose $f : \mathbb{R} \to \mathbb{R}$ is 1-1, onto and C^1 on $\mathbb{R}$ and g is its inverse function. Then g is C^1 on $\mathbb{R}$.

For the following questions give reasons and proofs for your claims. You may use theorems proved in class or in a textbook. Your grade will be based on your answers to at most 3 of the problems. Question 7 is worth 14 points, the other are worth 13 points each.

Question 4: Derive a necessary and sufficient condition (involving the coefficients b and c) for there to be a real solution of the equation

$$x^{2m} + bx + c = 0 \qquad \text{where m is a integer } \geq 1.$$

What is the maximum number of distinct real solutions that this equation can have? Why?

Question 5: Let $B = [0,1]^2$ be the closed unit square in the plane and $f : \mathbb{R}^2 \to \mathbb{R}$ be a G-differentiable function.

(a) Give a system of linear inequalities that describes B.

(b) What is the KKT system of equations and inequalities that hold at a local minimizer of f on B?

(c) If f attains a local minimum on B at the point $(0,1)$, what inequalities hold for the components of $\nabla f(0,1)$?

Question 6: Let C be a circular cylinder of radius R, height H and axis of symmetry along the z-axis. Under torsion, a point P with cylindrical polar coordinates (r, θ, z) is mapped to a point with Euclidean coordinates

$$(x_1, x_2, x_3) \;:=\; F(r, \theta, z) \;=\; (r\cos(\theta + \alpha z),\; r\sin(\theta + \alpha z),\; z)$$

Here $\alpha \in (0, 1)$. (a) Show that F maps C into itself and that F is 1-1.

(b) What is the inverse map of F?

(c) Evaluate the Jacobian (or G-derivative) $DF(r, \theta, z)$ and find its singular points.

Question 7: (14 points) Let $H \;:=\; L^2(0, 1)$ be the real Hilbert space that is the completion of the space $C[0, 1]$ w.r.t. the usual inner product. Let $I := [0, 1]$ and $K : I \times I \to \mathbb{R}$ be given by

$$K(x, y) \;:=\; \sum_{j=1}^{J} f_j(x)g_j(y) \qquad \text{where each } f_j, g_j \in C[I].$$

Assume that the $\{f_j : 1 \leq j \leq J\}$, $\{g_j : 1 \leq j \leq J\}$ are orthonormal in $L^2(I)$ and define the linear integral operator $\mathcal{K} : H \to H$ by

$$\mathcal{K}u(x) \;:=\; \int_0^1 K(x, y)u(y)\, dy$$

(i) Describe the range of $\mathcal{K}$.

(ii) Describe the null space of $\mathcal{K}$. Is it finite dimensional?

(iii) What is the adjoint operator of $\mathcal{K}$?

(iv) Find an upper bound for the L^2-operator norm of $\mathcal{K}$.

(v) Given $f \in H$, and that there are solutions $u \in H$ of the equation $\mathcal{K}u = f$, what can you say about f?

Question 8: Suppose $H = L^2(0, 1)$ as in the previous question and $k \in C[0, 1]$ obeys $0 \leq k(x) \leq M$ on $I := [0, 1]$. Define $\mathcal{K} : H \to H$ by

$$\mathcal{K}u(x) \;:=\; \int_0^x k(x - y)u(y)\, dy \qquad \text{for } x \in I.$$

(i) Find the norm of $\mathcal{K}$ as a map of H to itself.

(ii) What is the adjoint of the operator $\mathcal{K}$?

(iii) Define $\mathcal{K}^2 u := \mathcal{K}(\mathcal{K}u)$ for each $u \in H$. Find an explicit formula for the integral kernel of the operator $\mathcal{K}^2$.