Preliminary Examination Syllabus: Functions of a Real Variable

University of Houston, May 2024

For the preliminary exam, you are responsible for the topics, objects, concepts, and theorems listed below. You are expected to be able to **give precise definitions** of all objects and concepts listed here, to be familiar with their **basic properties and results**, and to be able to **give precise statements** of all theorems. You should be able to **use the theorems** (explaining why they apply), to **give standard counterexamples** demonstrating how various results fail when one or more hypotheses is removed, and to recall **standard examples** that appeared in lecture to demonstrate various concepts.

The primary text for the course was “Real Analysis for Graduate Students” by Richard Bass; the topics below correspond to Chapters 1–19 of that book. The textbooks by Axler and Folland are also good references, as is Rudin’s “Real and Complex Analysis”.

1. Background

Metric spaces: basic topology including limits, continuity, sequences, series, uniform convergence, open, closed, complete, compact. Complex numbers. Linear algebra.

2. Measures

- Algebra, σ-algebra, measurable space, measurable set. Borel σ-algebra.
- Measures: σ-finite, finite, probability, complete, Borel.
- Carathéodory extension theorem: premeasure on an algebra, construction of an outer measure μ^*, the σ-algebra of μ^*-measurable sets. Lebesgue and Lebesgue–Stieltjes measures. Null sets including Cantor set.
- Monotone class theorem, product σ-algebra, product measure.
- Signed measure, positive and negative sets. Hahn and Jordan decompositions. Total variation measure. Complex measures.
- Mutually singular measures, absolutely continuous measures, Cantor–Lebesgue function.

3. Integration

- Measurable functions. Approximation by simple functions.
- Lebesgue integral: non-negative functions, measurable functions, integrable functions.
- Monotone convergence theorem, Fatou’s lemma, dominated convergence theorem.
- Almost everywhere equality, convergence. Other types of convergence: in measure, in L^p, uniform, in L^∞. Implications and counterexamples.
- Chebyshev’s inequality. Egorov’s theorem, Luzin’s theorem, Fubini–Tonelli theorem.

4. Differentiation

- Radon–Nikodym theorem, Lebesgue decomposition theorem.
- Locally integrable functions on $\mathbb{R}^n$ are equal to local averages Lebesgue-a.e.: Vitali covering lemma, Lebesgue density theorem.
- Bounded variation, Lipschitz continuity, absolutely continuous functions. BV functions are the difference of increasing functions, hence a.e.-differentiable. Fundamental theorem of calculus for AC functions.

5. Basics of functional and Fourier analysis

- L^p spaces: conjugate exponents, basic inequalities (Young, Hölder, Minkoswki). Density of simple functions. Convolution, mollification, density of C_c^∞ in L^p.
- Normed vector spaces, dual space, duals of L^p spaces. Hahn–Banach theorem and its consequences, isometric embedding $X \to X^{**}$.
- Fourier transform on L^1, properties, Riemann–Lebesgue lemma, Schwartz class, Fourier inversion formula, Plancherel's theorem, extension of Fourier transform to L^2. Fourier series using Hilbert spaces.
- Riesz representation theorem for $C(X)^*$. Regularity of measures.
- Banach spaces: L^p, ℓ^p, C, C^r, C^α, $L(X,Y)$. Separability. Baire category theorem and its consequences: uniform boundedness principle (Banach–Steinhaus theorem), open mapping theorem, closed graph theorem.
- Hilbert spaces: inner product, Cauchy–Schwarz, parallelogram law, polarization identity, orthogonal complements and projections, Riesz representation (relate H and H^*), orthonormal sets, Gram–Schmidt procedure, Bessel's inequality, completeness, Parseval's identity, orthonormal basis.